Home range estimation under a restricted sampling scheme.

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Abstract

The analysis of animal movement has gained attention recently, and new continuous-time models and statistical methods have been developed. All of them are based on the assumption that this movement can be recorded over a long period of time, which is sometimes infeasible, for instance when the battery life of the GPS is short. We prove that the estimation of its home range improves if periods when the GPS is on are alternated with periods when the GPS is turned off. This is illustrated through a simulation study, and real life data. We also provide estimators of the stationary distribution, level sets (which provides estimators of the core area) and the drift function.

1. Introduction

Home range estimation is a major problem in animal ecology. It was defined by Burt (1943) as "the area traversed by the individual in its normal activities of food gathering, mating, and caring for young". Following this, there has arisen a considerable literature on the subject, see for instance the reviews in Worton (1987) or Powell (2000). Several models have been proposed to analyse the home range as well as the dynamics of animals. The first one (see Hayne (1949)) assumed that the available data is a sequence of locations recorded at some times, and estimates the home range by means of the convex hull of the points. In general, this overestimates the home range. Some proposals have been introduced to address this overestimation, for instance the use of "local convex hulls" (see Getz and Wilmers (2004)) or the r-convex hull (see Burgman and Fox (2003)). Some others are focused on the estimation of the co-called "utilization distribution" (the density function that describes the probability of finding the animal at a particular location). Among these, there is the so-called Brownian bridge model (BBMM), a parametric model, which was introduced in Horne et al. (2007). In the BBMM, given two recorded locations $a, b \in \mathbb{R}^2$ of the individual, the trajectory is interpolated by using a Brownian bridge $Z_t^{a,b,T}$ joining them (see Benhamou (2011), Buchin et al. (2012), Horne et al. (2007)), where $Z_0^{a,b,T} = a$, $Z_T^{a,b,T} = b$ and at time $t \in [0,T]$, $Z_t^{a,b,T} \sim N(\mu(t), \sigma^2(t) Id)$,

$$\mu(t) = a + t(b-a)/T, \qquad \sigma^2(t) = t(T-t)\sigma_m^2/T,$$

I_d is the 2×2 identity matrix and σ_m^2 is a diffusion coefficient that characterizes the animal's movement. Assuming that a set of recorded points $(a_i, b_i)_{i=1,\dots,n}$ is available, the goal of this model is to estimate the probability density function of the process Z_t (the position of the animal at time t), based on a mixture model of the random processes $\{Z_t^{a_i,b_i,T}\}_{i=1,\dots,n}$.

The BBMM can be unrealistic in some cases: for instance, if the irregularities of the terrain prevent the animal from visiting certain areas, the BBMM can assign no null probability to those areas, since no restriction is imposed on the Brownian bridge interpolating the recorded positions.

Avoiding this problem, other more general and flexible models assume that there is a continuous time reflected diffusion (RBMD) that governs the process, and the core-area of the species is estimated as well as its home range (see Cholaquidis et al. (2016, 2020)), together with the stationary distribution of the process.

A problem that appears in practice is that these models assume that it is possible to continuously record the location over a long period of time. This depends on the size of the animal (large animals allow a GPS with a much longer battery life). To overcome this problem, we propose recording the position of the animal continuously, but only for certain periods of time (of fixed length) in which the GPS is on (see Figure 1). More precisely, the GPS is set alternatively on during p intervals of length δ_1, and off during $p-1$ intervals of length δ_2.

This will be called the on-off model (see Section 3). Its existence and uniqueness is inherited from the RBMD. Even though there is a loss in the amount of available data, we obtain the same rate of convergence, in Hausdorff distance, as the one obtained in Cholaquidis et al. (2016). Moreover, we show that one gets a significant improvement using the on-off model in the following sense: if the GPS only transmits the location of the animal during p intervals of time δ_1, the upper bound for the Hausdorff distance between the estimators based on the on-off model and the support S is significantly smaller than for the case in which the GPS is turned on the same amount of time $p\delta_1$, but only between 0 and $p\delta_1$, see Theorem 1 below.

Next we consider the case where the set is r-convex (see the definition in next section) and provide convergence rates with respect to the Haussdorff

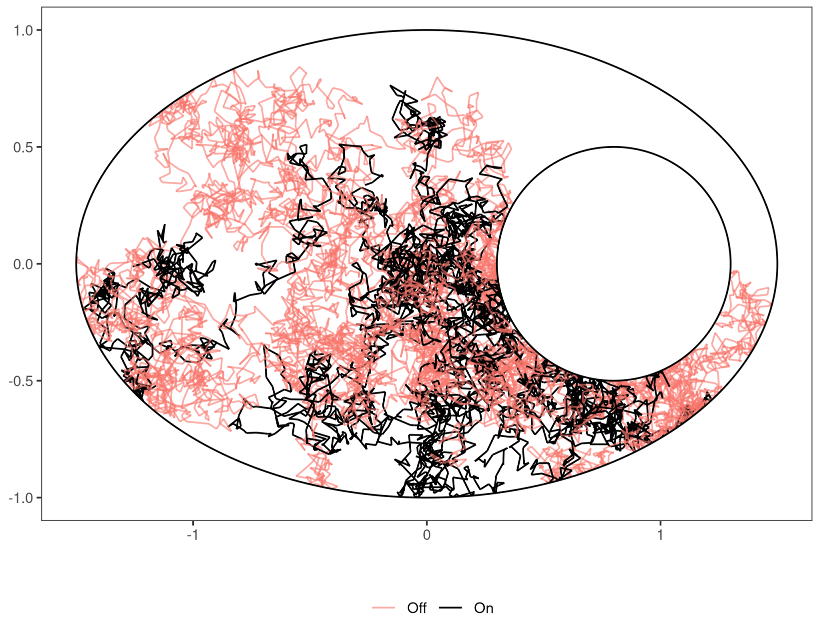

Figure 1: A trajectory of the on-off model is shown. The whole trajectory consist of 10^4 steps. In the first 250 steps the GPS is on, in the following 500 steps it is off, and so on. In black is the trajectory at times in which the GPS is on, in red is shown the trajectory at times which it is off.

distance and to the distance in measure for the r-convex hull. We also provide estimators of the stationary distribution, level sets and the drift function. Lastly we provide some simulation results and a real data example is analysed.

2. Some Preliminaries

The following notation will be used throughout the paper.

Given a set $S \subset \mathbb{R}^d$, we will denote by $\text{int}(S)$, $\overline{S}$ and ∂S the interior, closure and boundary of S, respectively, with respect to the usual topology of $\mathbb{R}^d$. The Borel σ-algebra in S will be denoted by $\mathcal{A}(S)$.

The parallel set of S of radius ε will be denoted by $B(S, \varepsilon)$, that is, $B(S, \varepsilon) = \{y \in \mathbb{R}^d : \inf_{x \in S} \|y - x\| \leq \varepsilon\}$. If $A \subset \mathbb{R}^d$ is a Borel set, then $\mu(A)$ denotes its d-dimensional Lebesgue measure. We will denote by

$B(x,\varepsilon)$ the closed ball in $\mathbb{R}^d$, of radius ε, centred at x, and $\omega_d = \mu(B(0,1))$. The open ball is denoted by $\mathring{B}(x,\varepsilon)$. Given two compact non-empty sets $A, C \subset \mathbb{R}^d$, the *Hausdorff distance* or *Hausdorff–Pompei distance* between A and C is defined by

$$d_H(A,C) = \inf\{\varepsilon > 0 : \text{such that } A \subset B(C,\varepsilon) \text{ and } C \subset B(A,\varepsilon)\}.$$

Given two measurable sets $A, C \subset \mathbb{R}^d$, the distance in measure between A and C is defined by

$$d_\mu(A,C) = \mu(A \setminus C) + \mu(C \setminus A).$$

The notion of r-convex sets, a well-known shape restriction in set estimation (see for instance Walther (1997, 1999)), extends convex sets to a much more flexible family of sets. It just replaces the hyperplanes in the definition of convex sets by the complements of balls of radius r, providing a very flexible class of sets.

Definition 1. A set $S \subset \mathbb{R}^d$ is said to be r-convex, for $r > 0$, if $S = C_r(S)$, where

$$C_r(S) = \bigcap_{\left\{\mathring{B}(x,r):\ \mathring{B}(x,r)\cap S=\emptyset\right\}} \left(\mathring{B}(x,r)\right)^c, \tag{1}$$

is the r-convex hull of S.

2.1 A brief outline of our theoretical results

- We prove that any set containing the trajectory of the on-off model (that is, the times at which the GPS is on) is a consistent estimator, in Hausdorff distance, of the home range and show the improvement we can attain with the proposed new model in Theorem 1.

- However, this is not the case if we want to estimate the set with respect to the distance in measure (w.r.t. Lebesgue measure). In this case, we propose to use the r-convex hull of the same trajectory of the on-off model. When S is r-convex, a natural estimator of S from a random sample $\mathcal{X}_n$ of points (drawn from a distribution with support S) is $C_r(\mathcal{X}_n)$. See, for instance, Rodríguez-Casal (2007); Pateiro-López and Rodríguez-Casal (2009). We show the convergence in the Hausdorff metric, while the convergence in measure is derived from Corollary 1 as mentioned in Remark 2.

- In order to estimate the core area (as a level set of the stationary density) and the drift component of the stochastic differential equation (2), we prove the uniform convergence of a kernel estimator of the stationary density, and derive estimators of the level sets in Theorem 2.
- An estimator of the the drift component is derived from the kernel density estimator by using Green's formula.
- In all cases we get almost sure convergence rates.

2.2 Reflected Brownian Motion with drift

Now we will give a brief review of the definition an main properties of the RBMD. The details can be found for instance in Cholaquidis et al. (2020) and references therein. In what follows, D is a bounded domain in $\mathbb{R}^d$ (that is, a bounded connected open set) such that ∂D is C^2. Given a d-dimensional Brownian motion $\{B_t\}_{t\geq 0}$ departing from $B_0 = 0$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}_x)$, and a function $f : \overline{D} \to \mathbb{R}$, the RBMD is the (unique) strong solution to the following reflected stochastic differential equation on $\overline{D}$ whose drift is given by the gradient of f,

$$X_t = X_0 + B_t - \frac{1}{2}\int_0^t \nabla f(X_s)ds + \int_0^t \mathbf{n}(X_s)\xi(ds), \qquad \text{where } X_t \in \overline{D},\ \forall t \geq 0. \tag{2}$$

Here we assume that ∇f is Lipschitz, while $\mathbf{n}(x)$ denotes the inner unit vector at the boundary point $x \in \partial D$. The term $\{\xi_t\}_{t\geq 0}$ is the corresponding *local time*, that is, a one-dimensional continuous non-decreasing process with $\xi_0 = 0$ that satisfies

$$\xi_t = \int_0^t \mathbb{I}_{\{X_s \in \partial D\}} d\xi_s.$$

Since we have assumed that ∂D is C^2, we can ensure that the geometric conditions for the existence of a solution of Equation (2), as required in Saisho (1987), are satisfied. We then get from Theorem 5.1 in Saisho (1987) that there exists a unique strong solution of the Skorokhod stochastic differential equation (2). The solution is a strong solution in the sense of definition 1.6 in Ikeda and Watanabe (1981). In Cholaquidis et al. (2020) it is proved that the RBMD given by (2) is non-trap, which is defined as follows.

Definition 2. We say that D is a *trap domain* for the stochastic process $\{Z_t\}_{t\geq 0}$ if there exists a closed ball $B \subset D$ with positive radius such that $\sup_{x\in D} \mathbb{E}_x T_B = \infty$, where $\mathbb{E}_x$ denotes the expectation w.r.t. $\mathbb{P}_x$. Otherwise D is called a *non-trap domain*.

The following proposition proven in Cholaquidis et al. (2020) will be used to get the consistency in Hausdorff distance, of the trajectory, as an estimator of the home range of the on-off model.

Proposition 1. *Let $D \subset \mathbb{R}^d$ be a bounded domain such that ∂D is C^2. Denote by π the invariant distribution of $\{X_t\}_{t\geq 0}$. If D is a non-trap domain for $\{X_t\}_{t\geq 0}$, then there exist positive constants α and β such that*

$$\sup_{x\in D} \left\| \mathbb{P}_x(X_t \in \cdot) - \pi(\cdot) \right\|_{TV} \leq \beta e^{-\alpha t}.$$

Here $\|\mu\|_{TV}$ stands for the total variation norm of the measure μ.

3. The On-Off Model

Our on-off model is defined as follows:

Definition 3. Given

- $S \subset \mathbb{R}^d$ a compact set
- $\{X_t : t > 0\}$ a reflected Brownian motion with drift, in S
- Two parameters $\delta_1, \delta_2 \in \mathbb{R}^+$
- A function $\{a_t : t > 0\}$ that varies over $\{0, 1\}$ intermittently, with $a_t = 1$ for periods of length δ_1 and $a_t = 0$ for periods of length δ_2. More precisely,

$$a_t = \sum_{k=0}^{\infty} \mathbb{I}_{[k(\delta_1+\delta_2),(k+1)\delta_1+k\delta_2]}(t).$$

We define the process

$$X_T^{ON} = \{X_t : t \in \mathcal{I}, t < T\}, \tag{3}$$

where $\mathcal{I} := \{t : a_t = 1\}$. Observe that the process X_T^{ON} is defined only on a union of disjoint intervals. The function a_t works like an "on-off" switch:

we only observe the process while $a_t = 1$ (i.e. the switch is 'On'), which happens on intervals of length δ_1, while it is not observable on intervals of length δ_2, in alternation.

The intuition behind this is that statistical properties should not vary so much when observing the full trajectory compared to when it is observed intermittently.

The following theorem gives some insight into the improvement obtained using the on-off model. It compares the upper bound for the Hausdorff distance between the trajectory of the on-off model and the support S, where the GPS is set on during p intervals of length δ_1 and is off during $(p-1)$ intervals of length δ_2, with the model in which the GPS is on the same amount of time $p\delta_1$ but only between 0 and $p\delta_1$.

Theorem 1. *Let $S \subset \mathbb{R}^d$ be a compact set such that $S = \overline{\operatorname{int}(S)}$ and ∂S is C^2. Let X_T^{ON} be defined as in Equation (3). Suppose the drift is a Lipschitz function given by the gradient of some function f, and assume that the stationary distribution π has density g. Write $c := \inf_{x \in S} g(x)$. Let S_T be any measurable set containing X_T^{ON}, such that $S_T \subset S$. Let $\varepsilon < 2(2\beta/c\omega_d)^{1/d}$, and assume that*

$$\delta_1 > \frac{1}{\alpha} \log\Big(\frac{2\beta}{c\omega_d(\varepsilon/2)^d}\Big). \tag{4}$$

Let $T = p\delta_1 + (p-1)\delta_2$. Then

$$\mathbb{P}\{d_H(S_T, S) > \varepsilon\} \leq C_1 \exp(-C_2 p\delta_1) \exp(-C_3 p). \tag{5}$$

However, if the GPS is on during the interval $[0, p\delta_1]$, the bound is

$$\mathbb{P}\{d_H(\widetilde{S}_{p\delta_1}, S) > \varepsilon\} \leq C_1 \exp(-C_2 p\delta_1) \tag{6}$$

where $\widetilde{S}_{p\delta_1}$ is any measurable set containing $X_{p\delta_1} = \{X_t : t < p\delta_1\}$, contained in S. The values of C_1, C_2, C_3 are given in the proof, they depend only on $\varepsilon, \mu(S), \beta, \alpha$ and ω_d. By (4) they are strictly positive.

Remark 1. The choice of the parameters is an important practical problem. Observe that given B the life time of the battery of the GPS, then $p = B/\delta_1$. So, given δ_1 and B, p is fixed. Equations (5) and (6) suggest choosing p as large as possible, and then δ_1 as small as possible, but fulfilling the condition in (4) together with $p\delta_1 = B$. This choice is consistent with the results we obtained in the simulations and the real-life data example.

The following corollary is a direct consequence of Theorem 1. It is proved following the same ideas used to prove Corollary 1 in Cholaquidis et al. (2016).

Corollary 1. *Under the hypotheses of Theorem 1, for any measurable set* S_T *containing* X_T^{ON} *we have*

a) $d_H(S_T, S) = o\big((\log(T)^2/T)^{1/d}\big)$ *a.s.*

b) If, moreover, S *and* S^c *are* r*-convex, for some* $r > 0$*, then* $d_H(C_r(X_T^{ON}), S) = o\big((\log(T)^2/T)^{1/d}\big)$ *a.s..*

Remark 2. The convergence in Hausdorff distance of a sequence of r-convex sets implies the convergence of its boundaries, as is proved in Theorem 3 in Cuevas et al. (2012), which, in turn, implies the convergence in measure, i.e., $d_\mu(C_r(\mathcal{X}_T), S) \to 0$, under the hypothesis of part b) of Corollary 1.

3.1 Estimation of the stationary distribution, level sets, and drift

In the stochastic differential equation (2) the drift $\nu(x)$ is given by the gradient ∇f of a function f, i.e., $\nu(x) = \frac{1}{2}\nabla f(x)$. By Green's formula, there exists a unique stationary distribution (for the RBMD and the on-off process X_T^{ON}) and it is given by $\pi(dx) = ce^{-f(x)}\mathbb{I}_D(x)dx := g(x)dx$, where c is a normalization constant. The density g can also be estimated using a kernel-based estimator

$$\hat{g}_n(x) = \frac{1}{nh_n^d}\sum_{i=1}^{n} K\Big(\frac{x - X_i}{h}\Big), \tag{7}$$

where $K : \mathbb{R}^d \to \mathbb{R}$ is a non-negative function, as proposed in Cholaquidis et al. (2020), based on a subsample of points at which the GPS is on. The a.s. uniform consistency of $\hat{g}_n$ is stated in the following theorem.

Theorem 2. *Under the hypotheses of Theorem 1, assume further that* g *is Lipschitz. Let* $\aleph_n = \{X_{(k+1)\delta_1 + k\delta_2} : k = 0, \dots, n-1\}$*. Let* $\hat{g}_n$ *be given by (7), based on* $\aleph_n$*. Assume that* K *is non-negative, Lipschitz and* $\int K(t)dt = 1$*. Let* $h = h_n \to 0$*,* $\beta_n \to \infty$*, and* $\alpha_n \to 0$ *such that* $\beta_n h_n \to 0$*,* $\alpha_n = o(1/\beta_n)$*,* $\log(n)/\beta_n \to 0$*, and* $\alpha_n nh^d/(\beta_n \log(n)) \to \infty$*. Then*

$$\beta_n \sup_{x\in S} |\hat{g}_n(x) - g(x)| \to 0 \quad a.s.$$

Moreover, if $\lambda > 0$ is such that $\partial G_g(\lambda) \neq \emptyset$ where $G_g(\lambda) = \{g > \lambda\}$, g is C^2 on a neighbourhood E of the level set λ and the gradient of g is strictly positive on E, then $d_H(\partial G_g(\lambda), \partial G_{\hat{g}_n}(\lambda)) = o(1/\beta)$ a.s.

The level sets will provide significant information about the time spent in those regions, in particular the core area will correspond to level sets with large values of λ.

An estimator of the drift function can also be derived from a plug-in method, and is given by $\hat{\nu}(x) = \frac{1}{2}\nabla \log(\hat{g}_n(x))$.

Figure 2 shows the estimated level sets for two different choices of δ_1 and δ_2, and the density g given by (8). The theoretical level sets are shown in Figure 3. The much better behaviour of the on-off model is clear when the number of points in the trajectory is small (2030 in the top panels), while the behaviour becomes similar when this number is large (98809 in the bottom panel).

The estimated density for the same set of parameters used in the first row of Figure 2 ($\delta_1 = 10$, $\delta_2 = 500$ with a total number of observations of $p\delta_1 = 2030$) is shown in Figure 4, where a Gaussian kernel with bandwidth $h = 2$ was employed.

3.2 Some simulation results

To simulate the RBMD we followed Cholaquidis et al. (2020): we first choose a step $h > 0$, and denote by $\text{sym}(z)$ the point symmetric with the point z with respect to ∂S. We start with $X_0 = x$ and suppose that we have obtained $X_i \in S$. To produce the following point, set

$$Y_{i+1} = X_i + Z_i + h\nabla f(X_i),$$

where Z_i is a centred Gaussian random vector, independent w.r.t. $Z_1, \dots, Z_{i-1}$, with covariance matrix $h \times (I_d)_{\mathbb{R}^2}$.

Then:

1. If $Y_{i+1} \in S$, set $X_{i+1} = Y_{i+1}$.
2. If $Y_{i+1} \notin S$ and $\text{sym}(Y_{i+1}) \in S$, set $X_{i+1} = \text{sym}(Y_{i+1})$.
3. If $Y_{i+1} \notin S$ and $\text{sym}(Y_{i+1}) \notin S$, set $X_{i+1} = X_i$.

Lastly, the on-off model is obtained from $X_1, \dots, X_N$ where we only keep those X_i such that $i \in \cup_{k=0}^{\infty}\{[k(\delta_1 + \delta_2)/h, (k+1)\delta_1/h + k\delta_2/h]\}$.

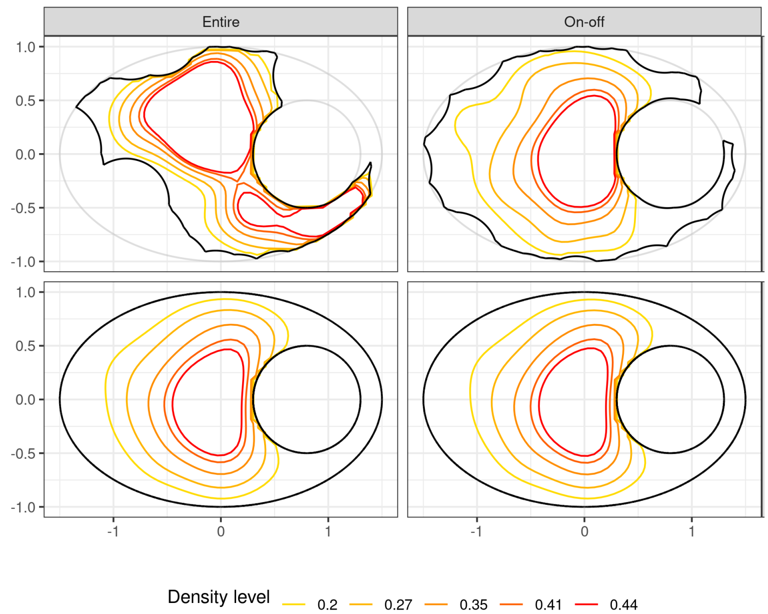

Figure 2: Top: Contour plot of the level sets of the estimated density function when $\delta_1 = 10$, $\delta_2 = 500$ with a total number of observations of $p\delta_1 = 2030$. Bottom: Contour plot of the level sets of the estimated density function when $\delta_1 = 500$, $\delta_2 = 10$ with a total number of observations of $p\delta_1 = 98809$. Left panels correspond to the entire trajectory, right panels to the on–off proposal.

We consider an RBMD in the set $S = E \setminus B((4/5, 0), 1/2)$, where $E = \{(x, y) \in \mathbb{R}^2 : 4x^2/9+y^2 \leq 1\}$, with drift function given by $\mu(x, y) = -(x, y)$. The stationary density is

$$g(x) = \frac{1}{c}e^{-(x^2+y^2)}\mathbb{I}_S(x, y) \quad \text{where } c = \iint_S \exp\left[-(x^2 + y^2)\right] dxdy. \tag{8}$$

The mean over 50 replications of the Hausdorff distance between S and the trajectory of the on-off model for different values of δ_1, δ_2, and h is shown in Table 1, for trajectories of $N = 10^5$ steps. In all tables we also report the median in parentheses. Figure 5 shows, in each of the 9 panels, for fixed δ_1/h, δ_2/h, the 50 Hausdorff distances (whose mean and median values are shown as solid and dashed lines), for $h \in \{0.01, 0.02, 0.03\}$.

A comparison of the on-off model and the model in which the GPS is on only between time 0 and time $p\delta_1$, stated in Theorem 1, is shown in Figure

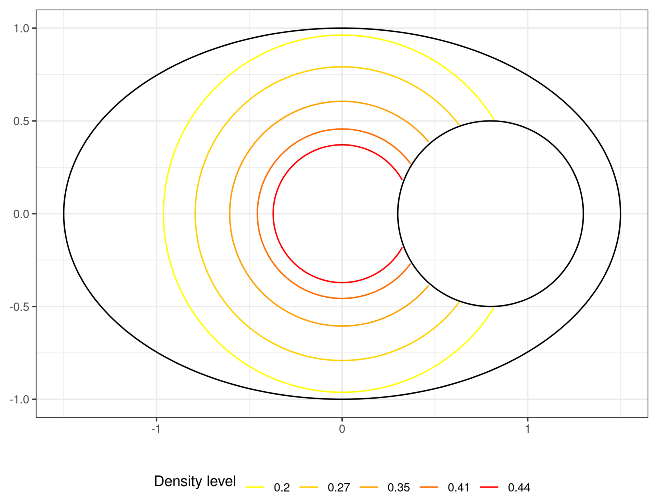

Figure 3: Theoretical level sets for the density g given by (8).

6 and Table 2. Figure 6 shows for each of the 9 panels previously considered (see Table 1) the 50 Hausdorff distances between the trajectory of the on-off model and S (square box). Each of them is joined with a segment, with the corresponding circle box (for the same trajectory), representing the Hausdorff distance between S and the same trajectory but where the GPS is on between 0 and $p\delta_1$.

Table 2 shows the average over 50 replications of the aforementioned distances. Table 3 shows the mean and median of the proportion of efficiency gain given by $1 - \frac{1}{50}\sum_{i=1}^{50} d_H(S_i^{ON}, S)/\frac{1}{50}\sum_{i=1}^{50} d_H(S_i^{[0:p\delta_1]}, S)$.

h	δ_1	δ_2: 100	250	500
0.001	100	0.1534 (0.1009)	0.2188 (0.1769)	0.2882 (0.2263)
0.001	250	0.1414 (0.0942)	0.1814 (0.1209)	0.2358 (0.1757)
0.001	500	0.1341 (0.0906)	0.1444 (0.0992)	0.1968 (0.1392)
0.002	100	0.0689 (0.0519)	0.1119 (0.0862)	0.1735 (0.1382)
0.002	250	0.0555 (0.0382)	0.0780 (0.0547)	0.1260 (0.0887)
0.002	500	0.0508 (0.0330)	0.0648 (0.0431)	0.0885 (0.0625)
0.003	100	0.0495 (0.0424)	0.0863 (0.0688)	0.1375 (0.1099)
0.003	250	0.0404 (0.0305)	0.0508 (0.0401)	0.0761 (0.0619)
0.003	500	0.0370 (0.0282)	0.0424 (0.0342)	0.0563 (0.0439)

Table 1: Mean and median over 50 replications of the Hausdorff distance, for the on-off model, with $N = 10^5$ steps.

h	δ_1	δ_2: 100	250	500
0.001	100	0.3072 (0.3128)	0.5151 (0.5134)	0.6478 (0.6726)
0.001	250	0.1983 (0.1143)	0.3090 (0.3128)	0.4603 (0.4726)
0.001	500	0.1469 (0.0811)	0.2043 (0.1143)	0.3072 (0.3128)
0.002	100	0.1214 (0.0784)	0.2580 (0.1476)	0.4047 (0.3165)
0.002	250	0.0800 (0.0435)	0.1215 (0.0784)	0.2178 (0.1216)
0.002	500	0.0674 (0.0368)	0.0826 (0.0453)	0.1215 (0.0784)
0.003	100	0.0632 (0.0520)	0.1260 (0.1026)	0.2774 (0.1668)
0.003	250	0.0426 (0.0352)	0.0632 (0.0520)	0.1006 (0.0849)
0.003	500	0.0376 (0.0286)	0.0473 (0.0368)	0.0632 (0.0520)

Table 2: Mean and median over 50 replications of the Hausdorff distance, for the trajectory observed in $[0, p\delta_1]$, with $N = 10^5$ steps.

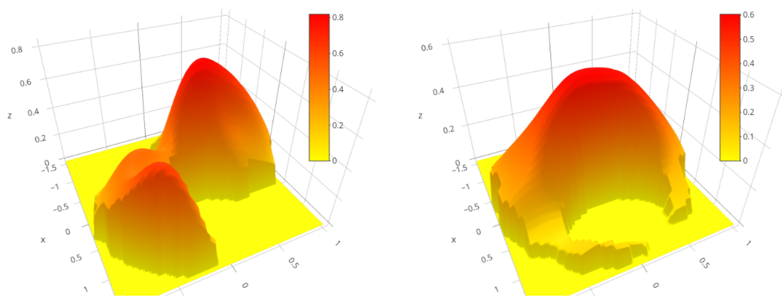

Figure 4: Estimated density using a Gaussian kernel with bandwidth $h = 0.2$. Right: the on-off model with $\delta_1 = 10$, $\delta_2 = 500$ with a total number of observations of $p\delta_1 = 2030$. Left: Estimated density based on trajectory observed with same number of observations but on $[0, 2030]$.

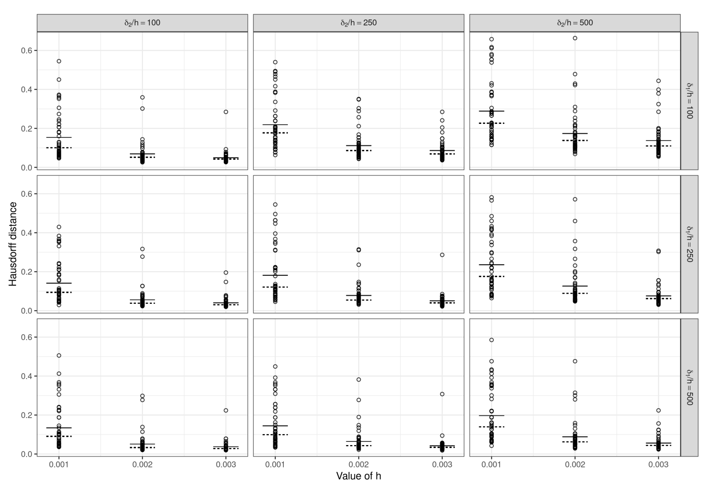

Figure 5: In each of the 9 panels we fixed δ_1/h, δ_2/h and plot the 50 Hausdorff distances for $h \in \{0.01, 0.02, 0.03\}$.

		δ_2		
h	δ_1	100	250	500
0.001	100	0.5008 (0.6775)	0.5753 (0.6554)	0.5551 (0.6635)
0.001	250	0.2866 (0.1760)	0.4129 (0.6136)	0.4878 (0.6282)
0.001	500	0.0874 (−0.1169)	0.2931 (0.1325)	0.3594 (0.5550)
0.002	100	0.4323 (0.3381)	0.5665 (0.4156)	0.5713 (0.5634)
0.002	250	0.3055 (0.1235)	0.3581 (0.3029)	0.4212 (0.2705)
0.002	500	0.2457 (0.1054)	0.2162 (0.0498)	0.2720 (0.2024)
0.003	100	0.2169 (0.1839)	0.3156 (0.3291)	0.5043 (0.3413)
0.003	250	0.0523 (0.1315)	0.1960 (0.2280)	0.2439 (0.2709)
0.003	500	0.0159 (0.0141)	0.1045 (0.0706)	0.1101 (0.1546)

Table 3: Mean and median of the proportion of efficiency gain given by $1 - \frac{1}{50}\sum_{i=1}^{50} d_H(S_i^{ON}, S)/\frac{1}{50}\sum_{i=1}^{50} d_H(S_i^{[0:p\delta_1]}, S)$.

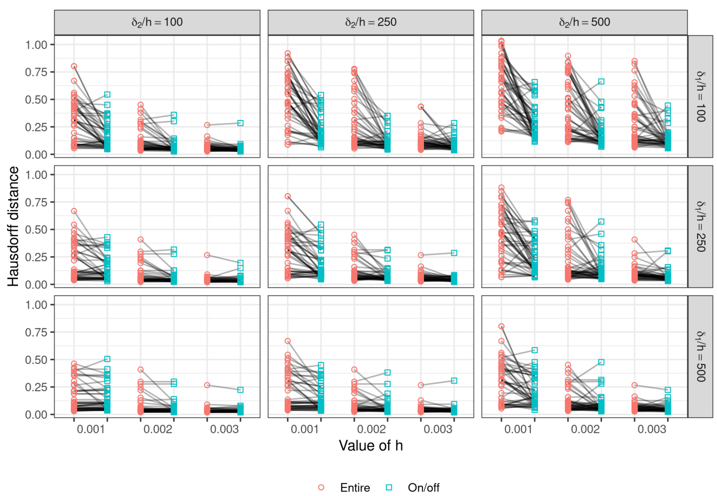

Figure 6: 50 Hausdorff distances between the trajectory of the on-off model and S (square box). Each of them is joined with a segment, with the corresponding circle box (for the same trajectory), representing the Hausdorff distance between S and the same trajectory but where the GPS is on between 0 and $p\delta_1$. For different values of δ_1 and δ_2.

h	δ_1	δ_2 100	250	500
0.001	100	0.0602 (0.0309)	0.1151 (0.0917)	0.2132 (0.2053)
0.001	250	0.0489 (0.0260)	0.0758 (0.0442)	0.1239 (0.1108)
0.001	500	0.0413 (0.0159)	0.0504 (0.0247)	0.0828 (0.0663)
0.002	100	0.0151 (0.0062)	0.0364 (0.0228)	0.0980 (0.0720)
0.002	250	0.0096 (0.0028)	0.0189 (0.0077)	0.0395 (0.0198)
0.002	500	0.0089 (0.0021)	0.0132 (0.0037)	0.0225 (0.0095)
0.003	100	0.0064 (0.0032)	0.0205 (0.0124)	0.0582 (0.0464)
0.003	250	0.0031 (0.0015)	0.0077 (0.0036)	0.0163 (0.0091)
0.003	500	0.0033 (0.0010)	0.0050 (0.0018)	0.0083 (0.0039)

Table 4: Mean and median over 50 replications of the distance in measure, for the on-off model, with $N = 10^5$ steps, and different values of h, δ_1 and δ_2.

3.3 The behaviour for the distance in measure

The same analysis is performed for the distance in measure. The mean over 50 replications for the distance in measure between the r-convex hull of the trajectory (for $r = 0.4$) and the set, for the same set of parameters used for the Hausdorff distance, are reported in Table 4 for the on-off model, for the trajectory only between 0 and $p\delta_1$, in Table 5. In Table 6 the relative efficiency is reported.

Figure 7 shows for each of the 9 panels previously considered the 50 measure distances between the 0.4-convex hull of the trajectory of the on-off model and S (square box). Each of them is joined with a segment, with the corresponding circle box (for the same trajectory), representing the measure distance between S and the same trajectory but where the GPS is on between 0 and $p\delta_1$.

		δ_2		
h	δ_1	100	250	500
0.001	100	0.1514 (0.1459)	0.3463 (0.3485)	0.6047 (0.5812)
0.001	250	0.0783 (0.0404)	0.1522 (0.1452)	0.2762 (0.2660)
0.001	500	0.0501 (0.0160)	0.0840 (0.0450)	0.1519 (0.1463)
0.002	100	0.0383 (0.0141)	0.1211 (0.0655)	0.2521 (0.2289)
0.002	250	0.0192 (0.0039)	0.0382 (0.0123)	0.0924 (0.0419)
0.002	500	0.0157 (0.0025)	0.0202 (0.0051)	0.0385 (0.0131)
0.003	100	0.0112 (0.0047)	0.0418 (0.0251)	0.1483 (0.0895)
0.003	250	0.0043 (0.0015)	0.0110 (0.0044)	0.0271 (0.0158)
0.003	500	0.0036 (0.0010)	0.0053 (0.0021)	0.0112 (0.0047)

Table 5: Mean (and median in parentheses) over 50 replications of the distance in measure, for the trajectory observed in $[0, p\delta_1]$, with $N = 10^5$ steps, and different values of h, δ_1 and δ_2.

		δ_2		
h	δ_1	100	250	500
0.001	100	0.6027 (0.7882)	0.6677 (0.7369)	0.6475 (0.6468)
0.001	250	0.3750 (0.3564)	0.5020 (0.6957)	0.5516 (0.5834)
0.001	500	0.1762 (0.0094)	0.4001 (0.4505)	0.4551 (0.5471)
0.002	100	0.6051 (0.5580)	0.6996 (0.6514)	0.6110 (0.6853)
0.002	250	0.4987 (0.2901)	0.5051 (0.3715)	0.5726 (0.5281)
0.002	500	0.4330 (0.1598)	0.3478 (0.2714)	0.4166 (0.2728)
0.003	100	0.4270 (0.3285)	0.5110 (0.5073)	0.6072 (0.4819)
0.003	250	0.2866 (−0.0015)	0.2970 (0.1785)	0.3975 (0.4236)
0.003	500	0.0786 (−0.0903)	0.0529 (0.1464)	0.2543 (0.1551)

Table 6: Mean and median of the proportion of efficiency gain given by $1 - \frac{1}{50}\sum_{i=1}^{50} d_\mu(S_i^{ON}, S)/\frac{1}{50}\sum_{i=1}^{50} d_\mu(S_i^{[0:p\delta_1]}, S)$.

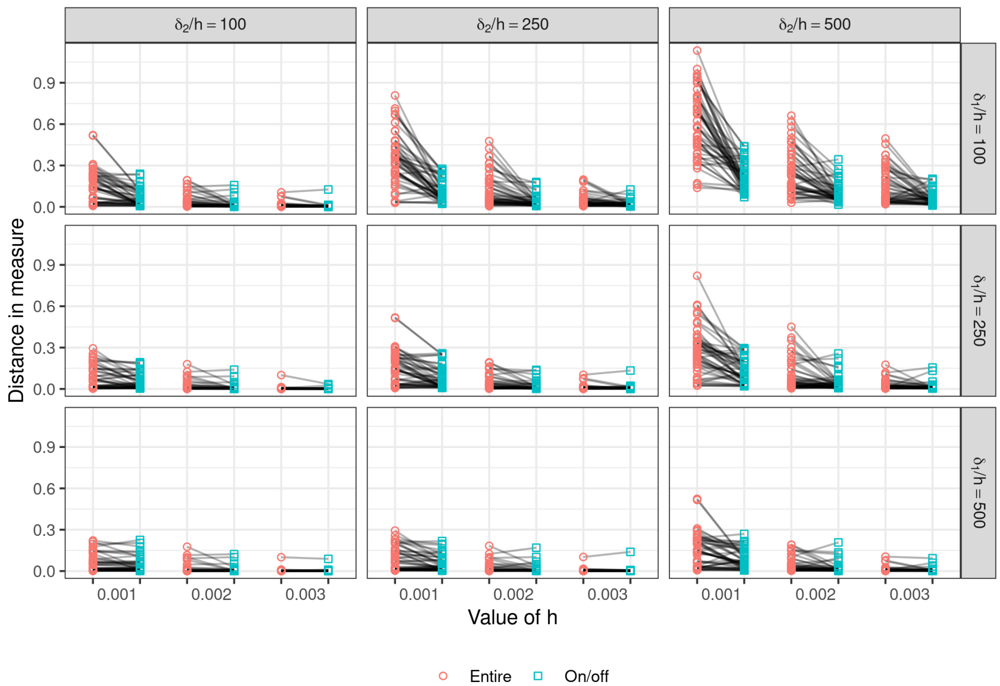

Figure 7: 50 distances in measure between the trajectory of the on-off model and S (square box). Each of them is joined with a segment, with the corresponding circle box (for the same trajectory), representing the distance between S and the same trajectory but where the GPS is on between 0 and $p\delta_1$. For different values of δ_1 and δ_2.

3.4 A real data example

In this section we demonstrate the performance of the on-off model using an example of real data. We consider a data set consisting of 1577 recorded positions of elephants in Loango National Park in western Gabon, available at the Movebank database. This dataset was also analysed in Cholaquidis et al. (2020). We first estimate the r-convex hull of this full trajectory. Later, we imagine that this full trajectory is not available at all, and we only have a subset of size $p\delta_1$ of the recorded locations. One approach is to observe the first $p\delta_1$ steps, and the other approach is considering our on-off strategy. Figure 8 shows, as a solid black line, the boundary of the 0.02-convex hull of the full trajectory, and the 0.02-convex hull under the two approaches.

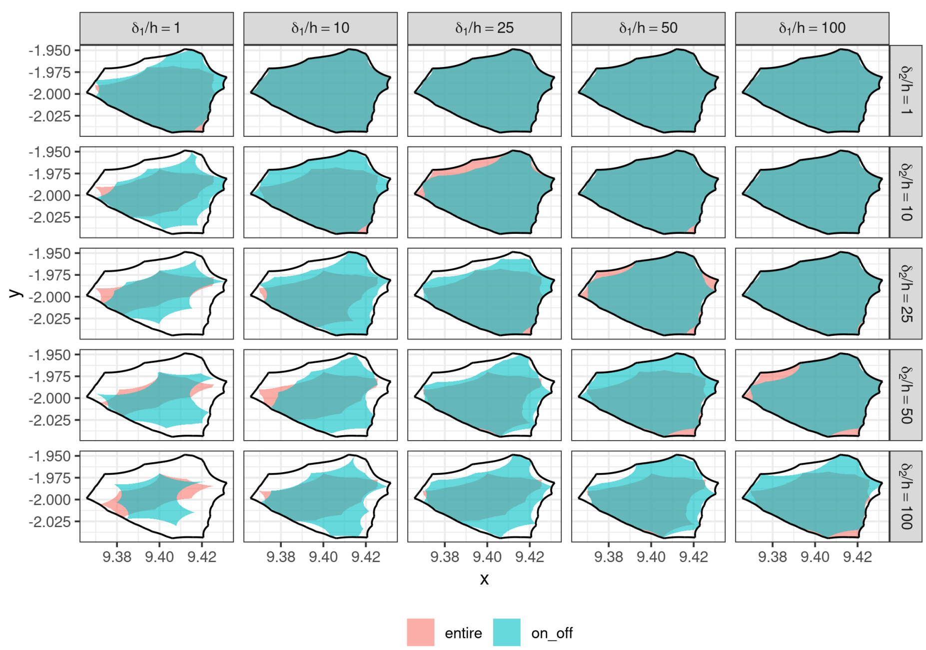

Figure 8: Each panel shows for different values of δ_1 and δ_2, in black solid line, the boundary of the 0.02-convex hull of the whole trajectory. In orange the 0.02 convex hull of the trajectory observed only between 0 and $p\delta_1$. Lastly, in green, the 0.02-convex hull of the trajectory of the on-off model.

4. Concluding Remarks

- We have shown, theoretically, through a simulation study, and through an example of real data, that the home range estimation problem under the reflected diffusion model is improved if instead of keeping the GPS on for the whole of the life time of the battery at once, the GPS is kept on intermittently.

- We obtain almost sure convergence rates for the estimation under the Haussdorf distance and the distance in measure.

- The stationary distribution can be estimated using a kernel type estimator as proposed in Cholaquidis et al. (2020).

- From the uniform convergence of the estimated stationary distribution, we derive estimators of the level sets, which can determine the core-area of the animals' home range.

- An estimator of the drift function can be derived from an estimator of the stationary distribution by a simple plug-in rule.

- Although a optimal choice of the parameters δ_1, δ_2 remains an open problem, the simulations confirm the assertion given in Remark 1 regarding the choice of the parameters: the best possible efficiency is obtained for small values of δ_1.

5. Appendix

5.1 Proof of Theorem 1

Define $\delta = c\omega_d(\varepsilon/2)^d/2$, $n = \left\lfloor \frac{T}{\frac{1}{\alpha}\log\frac{\beta}{\delta}} \right\rfloor$, and $t_i = \frac{i}{\alpha}\log\frac{\beta}{\delta}$ for $i = 1, \ldots, n$. Note that the condition for ε guarantees that $\beta/\delta > 1$. (Roughly speaking, $t_1, \ldots, t_n$ divide the interval $[0, T]$ into n intervals of length $\frac{1}{\alpha}\log\frac{\beta}{\delta}$.) Assuming that at time T the observation period of the process was completed, and taking p to be the number of periods of observations, $\delta_1 = l_1(t_i - t_{i-1})$ and $\delta_2 = l_2(t_i - t_{i-1})$ for some integers l_1 and l_2. We obtain that $T = p\delta_1 + (p-1)\delta_2$, and $n = p\,l_1 + (p-1)l_2$. We have $p\,l_1$ observations, so we have $p\,l_1 - 1$ transition probabilities of which $p-1$ occur between an on to off transition. We will assume that the process is ON in $[T-\delta_1, T]$.

Denote the $\varepsilon-$inner parallel set of S by

$$S^{(\varepsilon)} = \{x \in S : B(x, \varepsilon) \subset S\}.$$

Put $I_n := \{1, \ldots, n\} \cap \{i : a_{t_i} = 1\}$, the indices at which the process is observed. Then

$$\begin{aligned}\mathbb{P}\{d_H(S_T, S) > \varepsilon\} \leq & \mathbb{P}\{\exists x \in S^{(\varepsilon)} : \forall t \in \mathcal{I}, t < T : X_t \notin B(x, \varepsilon)\} \\ \leq & \mathbb{P}\{\exists x \in S^{(\varepsilon)} : \forall i \in I_n : X_{t_i} \notin B(x, \varepsilon)\}.\end{aligned}$$

Let $x_1, \ldots, x_N \in S^{(\varepsilon)}$ be such that $S^{(\varepsilon)} \subset B(x_1, \varepsilon/2) \cup \cdots \cup B(x_N, \varepsilon/2)$, and N is the smallest positive integer such that such covering of $S^{(\varepsilon)}$ is possible. $N = N(\varepsilon/2)$ is called the $\varepsilon/2$-covering number of $S^{(\varepsilon)}$. It is easy to see (and well known) that $N \leq \mu(S)/\mu(B(0, \varepsilon/4))) = (\varepsilon/4)^{-d}\mu(S)/\omega_d$.

If for some $x \in S$ we have $X_{t_i} \notin B(x, \varepsilon)$ for all $i \in I_n$, then there exists a $j \in \{1, \ldots, N\}$ such that $X_{t_i} \notin B(x_j, \varepsilon/2)$ for all $i = 1, \ldots, n$. Thus, continuing the chain of inequalities above,

$$\begin{aligned}\mathbb{P}\{d_H(S_T, S) > \varepsilon\} \leq & \mathbb{P}\{\exists j \in \{1, \ldots, N\} : \forall i \in I_n : X_{t_i} \notin B(x_j, \varepsilon/2)\} \\ \leq & N \sup_{x \in S^{(\varepsilon)}} \mathbb{P}\{\forall i \in I_n : X_{t_i} \notin B(x, \varepsilon/2)\}.\end{aligned}$$

Next we estimate the probability on the right-hand side. Recall that the process is ON in $[T - \delta_1, T]$, then $n \in I_n$. For all $x \in S^{(\varepsilon)}$,

$$\begin{aligned}&\mathbb{P}\{\forall i \in I_n : X_{t_i} \notin B(x, \varepsilon/2)\} \\ &\quad= \mathbb{P}\{X_{t_n} \notin B(x, \varepsilon/2) | \forall i \in I_{n-1} : X_{t_i} \notin B(x, \varepsilon/2)\} \\ &\qquad \times \mathbb{P}\{\forall i \in I_{n-1} : X_{t_i} \notin B(x, \varepsilon/2)\} \\ &\quad= \mathbb{P}\{X_{t_n} \notin B(x, \varepsilon/2) | X_{t_{n-1}} \notin B(x, \varepsilon/2)\} \\ &\qquad \times \mathbb{P}\{\forall i \in I_{n-1} : X_{t_i} \notin B(x, \varepsilon/2)\} \\ &\qquad \text{(since } X_t \text{ is a Markov process)}\end{aligned}$$

Denote by π the invariant distribution of the process $\{X_t^{ON}\}_{t>0}$. Let us iterate this process the last l_1 steps at which the process is ON. Then

$$\begin{aligned}&\mathbb{P}\{\forall i \in I_n : X_{t_i} \notin B(x, \varepsilon/2)\} = \\ &\quad= \mathbb{P}\{X_{t_n} \notin B(x, \varepsilon/2) | X_{t_{n-1}} \notin B(x, \varepsilon/2)\} \times \cdots \times \\ &\mathbb{P}\{X_{t_{n-l_1+2}} \notin B(x, \varepsilon/2) | X_{t_{n-l_1+1}} \notin B(x, \varepsilon/2)\} \times \mathbb{P}\{\forall i \in I_{n-l_1+1} : X_{t_i} \notin B(x, \varepsilon/2)\} \\ &= \mathbb{P}\{\forall i \in I_{n-l_1+1} : X_{t_i} \notin B(x, \varepsilon/2)\} \prod_{i=0}^{l_1-2} \mathbb{P}\{X_{t_{n-i}} \notin B(x, \varepsilon/2) | X_{t_{n-i-1}} \notin B(x, \varepsilon/2)\}\end{aligned}$$

Now, by Proposition 1 and the definition of δ ($\delta = c\omega_d(\varepsilon/2)^d/2$), for all $i = 0, \ldots, l_1 - 2$,

$$\begin{aligned}
\mathbb{P}\{X_{t_{n-i}} \notin B(x,\varepsilon/2) | X_{t_{n-i-1}} \notin B(x,\varepsilon/2)\} = \\
& 1 - \mathbb{P}\{X_{t_{n-i}} \in B(x,\varepsilon/2) | X_{t_{n-i-1}} \notin B(x,\varepsilon/2)\} \\
& \leq 1 - \pi(B(x,\varepsilon/2)) + \beta \exp\{-\alpha(t_{n-i} - t_{n-i-1})\} \\
& = 1 - \pi(B(x,\varepsilon/2)) + \delta \\
& \leq 1 - c\omega_d(\varepsilon/2)^d + \delta = 1 - c\omega_d(\varepsilon/2)^d/2
\end{aligned}$$

So

$$\prod_{i=0}^{l_1-2} \mathbb{P}\{X_{t_{n-i}} \notin B(x,\varepsilon/2) | X_{t_{n-i-1}} \notin B(x,\varepsilon/2)\} \leq (1 - c\omega_d(\varepsilon/2)^d/2)^{l_1-1}$$

Using a similar argument, we will bound the term $\mathbb{P}\{\forall i \in I_{n-l_1-1} : X_{t_i} \notin B(x,\varepsilon/2)\}$. Observe that now at time t_{n-l_1+1} the GPS is on but it is off at time t_{n-l_1}, and it is also off at times t_{n-l_1-j} for all $j = 0, \ldots, l_2 - 1$. This implies that $I_{n-l_1+1} = \{n - l_1 + 1\} \cup I_{n-l_1-l_2}$. Then

$$\begin{aligned}
& \mathbb{P}\{\forall i \in I_{n-l_1+1} : X_{t_i} \notin B(x,\varepsilon/2)\} = \\
& \mathbb{P}(X_{t_{n-l_1+1}} \notin B(x,\varepsilon/2) | X_{t_{n-l_1-l_2}} \notin B(x,\varepsilon/2)) \mathbb{P}\{\forall i \in I_{n-l_1-l_2} : X_{t_i} \notin B(x,\varepsilon/2)\} \leq \\
& \qquad (1 - c\omega_d(\varepsilon/2)^d + \beta e^{-\alpha\delta_2}) \mathbb{P}\{\forall i \in I_{n-l_1-l_2} : X_{t_i} \notin B(x,\varepsilon/2)\}
\end{aligned}$$

So, by iterating the argument, $\mathbb{P}\{\forall i \in I_n : X_{t_i} \notin B(x,\varepsilon/2)\}$ is bounded from above by

$$\begin{aligned}
& (1 - c\omega_d(\varepsilon/2)^d/2)^{p(l_1-1)} (1 - c\omega_d(\varepsilon/2)^d + \beta e^{-\alpha\delta_2})^{p-1} \\
& \leq \exp\{-(l_1 - 1)p\delta\} \exp\left\{-(p-1)(2\delta - \delta(\delta/\beta)^{l_2-1})\right\}
\end{aligned} \tag{9}$$

To obtain the gain of using the on-off model compared with the model in which the GPS is set on only between 0 and $p\delta_1$, note that

$$\exp\{-(l_1 - 1)p\delta\} = \exp\left\{-\frac{(l_1 - 1)pt_1}{t_1}\delta\right\},$$

and $(l_1 - 1)t_1 = \delta_1 - t_1$. Then if the process is on only on the interval $[0, p\delta_1]$ (that is, $l_2 = 0$),

$$\mathbb{P}\{d_H(\widetilde{S}_{p\delta_1}, S) > \varepsilon\} \leq \frac{(\varepsilon/4)^{-d}\mu(S)}{\omega_d} \exp\left\{-\frac{c\omega_d(\varepsilon/2)^d p(\delta_1 - \frac{1}{\alpha}\log\frac{\beta}{\delta})}{2\frac{1}{\alpha}\log\frac{\beta}{\delta}}\right\}$$
$$= C_1 \exp(-C_2 p\delta_1)$$

However, from (9) in the on-off model, the bound obtained is

$$\mathbb{P}\{d_H(S_T, S) > \varepsilon\} \leq \frac{(\varepsilon/4)^{-d}\mu(S)}{\omega_d} \exp\left\{-\frac{c\omega_d(\varepsilon/2)^d p(\delta_1 - \frac{1}{\alpha}\log\frac{\beta}{\delta})}{2\frac{1}{\alpha}\log\frac{\beta}{\delta}}\right\} \times$$
$$\times \exp\left\{-\delta(p-1)(2-(\delta/\beta)^{l_2-1})\right\} = C_1 \exp(-C_2 p\delta_1)\exp(-C_3 p)$$

5.2 Proof of Theorem 2

It is easy to prove, following the ideas used to prove Proposition 2 in Cholaquidis et al. (2020), that the chain $\aleph_n$ is geometrically ergodic. Then $\beta_n \sup_{x\in S} |\hat{g}_n(x) - g(x)| \to 0$ a.s. follows as a direct application of Theorem 1 in Cholaquidis et al. (2020). From Corollary 1 together with Remark 4 in Cholaquidis et al. (2020) it follows that $d_H(\partial G_g(\lambda), \partial G_{\hat{g}_n}(\lambda)) = o(1/\beta) \quad a.s.$.

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