Rawlsian Assignments*

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Abstract

We study the assignment of indivisible goods to individuals without monetary transfers. Previous literature has mainly focused on efficiency and *individually* fair assignments; consequently, egalitarian concerns have been overlooked. Drawing inspiration from the allocation of apartments in housing cooperatives—where families prioritize egalitarianism in assignments—we introduce the concept of Rawlsian assignment. We demonstrate the uniqueness, efficiency and anonymity of the Rawlsian rule. Our findings are validated using cooperative housing preference data, showing significant improvements in egalitarian outcomes over both the probabilistic serial rule and the currently employed rule.

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1 Introduction

We study the problem of allocating a set of indivisible goods to agents when monetary transfers cannot be used. The indivisibility of objects makes it generally impossible to achieve fairness from an ex-post perspective. As a remedy, we adopt an ex-ante perspective where each agent

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receives a random allocation, i.e., a lottery over the set of objects. Inspired by the Rawlsian concept of fairness, we propose a new solution concept, examine its properties, and evaluate its performance using real data.

Our primary motivation is the assignment of apartments in housing cooperatives. Housing cooperatives typically involve a group of families who participate in the construction of a building. Upon completion of the building, the problem of distributing the apartments among the families arises. The use of prices is, in general, not allowed by regulation, and the assignment process relies on the ordinal preferences of the families. In Uruguay, the regulation also states that apartments should be assigned randomly; however, participants have the option of using another procedure if they unanimously agree on it. For the Uruguayan housing cooperatives we study, the main concern of the members is the *egalitarianism* of the final assignment. They aim to avoid unequal situations where, for example, some families receive their most preferred apartments while others rank their assignments very low. The spirit of this principle is not exclusive to Uruguay; it is present in many other cases as well (for more instances, we refer the reader to www.housinginternational.coop). Another example where a similar concern may exist is the assignment of faculty offices to professors in a university department.

There have been many important contributions in the literature regarding the efficiency and envy-freeness of the assignments. However, little attention has been paid to *egalitarian* concerns. In this paper, we introduce a new concept to address this issue and examine its properties. Furthermore, using data from several housing cooperatives, we demonstrate that the solution we propose is more egalitarian from a Rawlsian perspective compared to the currently implemented one. Thus, the Rawlsian rule we recommend improves upon the outcomes achieved by the current rule.

We adopt Rawls' concept of justice and assess an assignment based on the well-being of the worst-off individuals. The rank of an object on an agent's preferences is the position of the object in her preferences. Thus, a higher rank indicates lower satisfaction (for example, the most preferred object has rank 1). The analysis begins by identifying, for a given assignment, who the worst-off agents are. If we were considering deterministic assignments, we could examine the rank of the assigned object. The worst-off agents would be those associated with the highest rank. Subsequently, we could select assignments that minimize the highest rank (i.e., maximize the *satisfaction* of the worst-off agents). In general, there are multiple assignments that satisfy this criterion, allowing for recursive application of the same criterion, as proposed by Sen (1970).

When we expand the analysis to include random assignments, defining the worst-off agents becomes more challenging. If agents' cardinal utilities were known, one could consider the expected utility of each agent and identify the agent with the lowest expected utility (after some normalization). However, in our problem, we only have access to agents' ordinal preferences. So instead, for each agent, we determine the rank of her least preferred object among those she receives with positive probability. We then focus on the agents for whom this rank is the highest. Among these agents, the worst-off individuals are those who receive the object with the highest probability.

To compare two assignments, we first consider the worst-off agent, that is, the one who receives her least preferred object with the highest probability. If the probabilities are the same under both assignments, we turn to the agent who receives her least preferred object with the second-highest probability. If all the probabilities associated with the least preferred object of each agent are the same, we conduct the same comparison with the two least preferred objects of each agent. If, at any point, the probability in the first assignment is strictly higher than the corresponding probability in the second assignment, we say that the first is Rawlsian-dominated by the second. An assignment is considered Rawlsian if it is not Rawlsian-dominated by any other assignment.

We first show that, in any problem, there always exists a **unique** Rawlsian assignment. Roughly, if there were two Rawlsian assignments, one would consider the average assignment. In this assignment, agents who receive (with positive probability) the objects associated with the highest rank would be assigned them with a lower probability.

We extend preferences over objects to preferences over lotteries using first order stochastic dominance (sd). In words, an agent sd-prefers a lottery if it guarantees her a weakly higher probability of receiving her most preferred object, a weakly higher probability of receiving her two most preferred objects, and so on. An assignment sd-dominates another assignment if for every agent, the lottery she receives in the first assignment is sd-preferred to the lottery in the second assignment. We prove that the Rawlsian rule is **sd-efficient**, it is not sd-dominated by any other rule, and **anonymous**, agents' allocations do not depend on their names. The drawback is that the Rawlsian rule is manipulable (it is not sd-strategyproof).

The construction of our concept resembles the welfarist definition of the classic probabilistic serial rule (Bogomolnaia and Moulin, 2001), proposed by Bogomolnaia (2015, 2018). The author describes the probabilistic serial (PS) rule in terms of the cumulative probabilities of each agent being assigned her top k objects. She shows that the PS rule is the unique rule that lexicographically maximizes the cumulative probabilities for all agent-preference pairs (i, k). The Rawlsian criterion is based on a similar procedure but starts from the least preferred objects and minimizes their probability. Moreover, the Rawlsian rule lexicographically minimizes the cumulative probabilities preference by preference, instead of lexicographically minimizing the entire vector of cumulative probabilities.

Next, we study how to compute the Rawlsian assignment for a given problem. We introduce an algorithm to compute it in polynomial time by solving at most $\frac{n^3+n^2}{2}$ linear programs.

We illustrate our analysis using a set of preferences from housing cooperatives in Uruguay. We compare our solution with the outcomes of the PS rule, and the rule which is currently used, called the **MTAV** (proposed by Prino, Sánchez, and Cancela (2016)). Roughly, the MTAV runs as follows. First, it selects all deterministic assignments that minimize the maximum rank. Second, among these assignments, it considers those that maximize the sum of the families' utilities, assuming that the utility of the apartment ranked in position k is $n-k$ for each family.

Finally, if multiple assignments are selected, one is randomly chosen (see Appendix 10.9 for a formal description).

The general finding is that the Rawlsian solution significantly improves the assignment for the least favored families. For instance, under the PS assignment, there is at least one family who receives their least preferred apartment with a positive probability in all but two out of 24 cooperatives. In contrast, with the Rawlsian assignment, only three cooperatives have a family that receives their least preferred apartment with positive probability. Although there are winners and losers when transitioning from the PS rule to the Rawlsian rule, the number of families who prefer their Rawlsian assignment is always larger. Regarding the MTAV, by construction, its maximum rank (the rank of the assigned apartment in the preferences of the worst-off family) coincides with that of the Rawlsian rule. However, there are cases where the Rawlsian rule assigns fewer families to their least preferred apartment.

Another measure to compare the assignments is the sum of the probabilities with which families are assigned apartments ranked in the first k positions, for each $k \in \{1, \dots, n\}$. We call it the expected number of families that are assigned apartments with rank less than or equal to k. The Rawlsian rule assigns a lower expected number of families their least preferred apartments compared to the PS rule. It also assigns a lower expected number of families their top choices, especially their first choice. The MTAV falls between these two distributions. Figure 1 displays the rank distribution graph for a cooperative with 28 families. Therefore, our solution provides an alternative to both the PS and MTAV that significantly improves upon them from an egalitarian perspective.

Finally, we complement the empirical findings by analyzing the maximum rank of the Rawlsian rule in large markets. We consider markets of size n, where agents' preferences are drawn i.i.d. from a uniform distribution, and study the limit of the expected maximum rank as n tends to infinity. Although the average maximum rank of the Rawlsian rule tends to infinity, we show that it grows at a slow rate (it is upper bounded by $\lfloor \ln(n) \rfloor$ plus a constant). In a market of size $n = 1000$, for example, our result implies that the expected rank of the least preferred assigned object is at most 9. For the PS rule, however, simulations suggest that the expected maximum rank in a random market of size n is very close to n.

In the next section, we place our contributions within the related literature. Section 3 presents the model and definitions. In Section 4, we define the concept of a Rawlsian assignment. Section 5 contains our main results and relates the Rawlsian assignment to other concepts in the literature. In Section 6, we describe an algorithm to find the Rawlsian assignment for a given problem, and in Section 7 we use it to illustrate the results with data from different housing cooperatives. Section 8 includes the analysis for large markets. We conclude in Section 9. All the omitted proofs are in the Appendix.

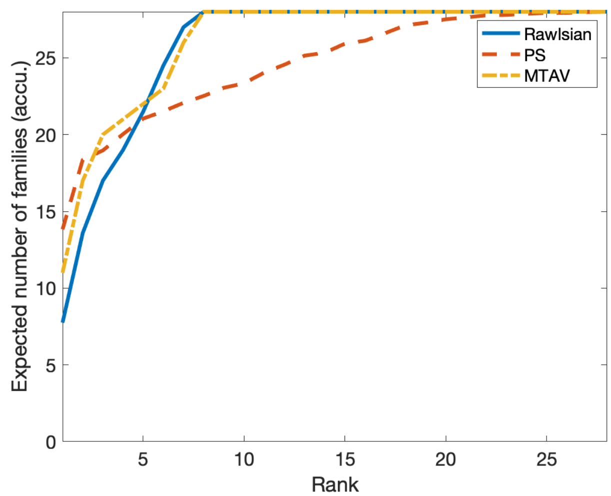


Figure 1: Rank distribution for cooperative C_5

2 Related Literature

Since the introduction of the assignment problem by Hylland and Zeckhauser (1979), there have been numerous significant contributions related to sd-efficiency, sd-envy-freeness, and sd-strategyproofness. Notable among these is the random serial dictatorship (or random priority) rule introduced by Zhou (1990) and Abdulkadiroğlu and Sönmez (1998). While this rule is sd-strategyproof (it is immune to individual preferences manipulation), its outcome may be stochastically dominated with respect to individual preferences, thus lacking sd-efficiency. In response, Bogomolnaia and Moulin (2001) proposed the PS rule, which is sd-efficient but not sd-strategyproof. Moreover, the PS rule is sd-envy-free: the lottery each agent receives stochastically dominates with respect to individual preferences the lottery of every other agent. The random serial dictatorship rule is not sd-envy-free. In this paper, we introduce the Rawlsian rule, which preserves the sd-efficiency of the PS rule while satisfying an egalitarian requirement.

The Rawlsian idea of justice, as described by Rawls (1971), has found applications in two-sided matching markets. For instance, Masarani and Gokturk (1989), Romero-Medina (2005), and Kuvalekar and Romero-Medina (2024) explore the compatibility of this concept of justice with stability in a marriage market. They adopt the Rawlsian criterion to select a stable matching that treats both sides of the market "symmetrically." In a school choice model, Galichon, Ghelfi, and Henry (2023) show that stability often comes at the cost of extreme forms of inequality. The assignment problem we study differs from the literature mentioned above. There are agents on one side and objects on the other, objects do not possess priorities, and the notion of stability does not apply. Klaus and Klijn (2010) adapt the Rawlsian criterion to the roommate problem and demonstrate its compatibility with stability. Another way to compare the inequality of different assignments is by applying the Lorenz order. This method has been used in our framework by various studies, such as Pycia and Ünver (2017) and Harless and Manjunath

(2018).

In the same framework as ours, Afacan and Dur (2024) study a Rawlsian notion of fairness but consider **only** deterministic assignments. They define an assignment to be more Rawlsian than another if the assignment ranking of the worst-off agent is lower in the former than in the latter; and in the case of equality, the size of the worst-off agents group is smaller in the former. For the case of deterministic assignments, our notion of Rawlsian assignment is a refinement of their notion. However, expanding the analysis to probabilistic assignments allows us to show, in contrast to Afacan and Dur (2024), the existence of a unique Rawlsian assignment that is sd-efficient. Ortega and Klein (2023) compare different rules in school choice in terms of their expected maximum rank. They show that the deferred acceptance and top trading cycles rules perform poorly on this measure, being outperformed by the rule that minimizes the sum of ranks for students. Ortega, Ziegler, and Arribillaga (2024) defines the Rawlsian inequality of an assignment as the ratio between the maximum rank of the assignment and that of the Rawlsian assignment. The authors compare the deferred acceptance rule and the efficiency-adjusted deferred acceptance rule of Kesten (2010) in terms of their Rawlsian inequality.

The study by Duddy (2022) is the closest to our paper. Duddy introduces a new fairness criterion which he calls the "egalitarian criterion", along with two rules that satisfy this criterion: the "positive equality" rule and the "prudent equality" rule. The positive equality rule uses the following order to compare assignments. It starts by considering, for each assignment, the probability with which each agent receives her most preferred object, and takes the lowest probability. The rule selects the assignment for which the lowest probability is the highest. If the two lowest probabilities are the same, it moves on to the agent who receives her two most preferred objects with the lowest total probability, and selects the assignment for which the lowest total probability is the highest, and so on. The positive equality rule chooses the maximum assignment based on this order. On the other hand, the prudent rule performs a similar comparison but starts by considering the agent who receives, at each assignment, her least preferred object with lowest probability. Then, it selects the assignment for which the lowest probability is the highest. *When the two assignments coincide, it then moves to the agent who receives her least and second-least preferred objects with the highest total probability, and so on.*

In contrast, the Rawlsian assignment begins by considering the agent who receives her least preferred object with the larger probability and aims to minimize this probability. However, in the case of ties, instead of considering the least and second-least preferred objects, it moves to the agent who receives her least preferred object with the second highest probability. Intuitively, our proposal focuses on the worst-off agent defined as the agent who receives her least preferred object with the larger probability, and then moves to the second worst-off agent. The Rawlsian assignment does not satisfy Duddy's egalitarian criterion, and there are egalitarian assignments that are not Rawlsian. Thus, the two concepts are independent. For a more detailed discussion, we refer to Appendix 10.5. Also related is the paper by Feizi (2022) who introduces a notion of inequality in random assignments. The key element of his definition is the number of agents who envy other agents in a representation of the assignment as a lottery over deterministic assignments. The concept aims to minimize this envy which is different from our approach.

The notion of Rawlsian assignments is also related to the downward lexicographic extension of stochastic dominance (Cho and Doğan, 2016; Cho, 2018). Different from Cho (2018), who compares two lotteries from an individual perspective, we use a similar criterion but apply it to the whole assignment. Also, Cho and Doğan (2016) show that upward lexicographic (ul) efficiency and sd-efficiency are equivalent. A Rawlsian assignment is sd-efficient (Proposition 3), and thus, also ul-efficient, but not the other way around. Finally, other papers have also used the rank distribution to define different concepts. Featherstone (2020) introduces rank efficiency. An assignment is rank efficient if its rank distribution cannot be stochastically dominated. We prove in Appendix 10.4 that there is no relation between Rawlsian and rank efficient assignments.

3 Primitives and definitions

Let $I = \{1, \dots, n\}$ be the set of agents and O the set of objects (with $|O| = n$). Each agent $i \in I$ has (strict) preferences over the set of objects, denoted by $\succeq_i$. A preference profile is denoted by $\succeq = (\succeq_i)_{i \in I}$. Sometimes we represent agent i's preferences as an n-dimensional vector $r_i \in \{1, \dots, n\}^n$, where $r_{io} = k$ means that object o is ranked k-th by agent i. The assumption of strict and complete preferences fits our main motivation: families in the cooperatives have to rank all the apartments, and ties in preferences are not allowed. An assignment problem is a tuple $(I, O, \succeq)$. We fix the sets of agents and objects, and define a problem as a preference profile $\succeq$.

A solution to an assignment problem is a (random) assignment $x = (x_i)_{i \in I}$, where each x_i is a probability distribution over O, and for every object $o \in O$, $\sum_{i \in I} x_{io} = 1$. We interpret $x_{io} \in [0, 1]$ as the probability with which agent i is allocated object o. An assignment is deterministic if all of its entries are either zero or one. The Birkhoff-von Neumann theorem (Birkhoff, 1946; von Neumann, 1953) states that any random assignment can be written as a convex combination of deterministic assignments.

We usually describe an assignment by a matrix where the rows are indexed by the agents and the columns are indexed by the objects, and for each i, row i is the lottery agent i receives. Let X be the set of assignments (or equivalently, the set of $n \times n$ bi-stochastic matrices, that is, matrices of non-negative real numbers in which the entries in each row sum to 1, and the entries in each column sum to 1). A **rule** is a function ϕ which maps every problem to an assignment: for every $\succeq$, $\phi(\succeq) \in X$. We will refer to the lottery assigned to agent i – given by the i-th row of an assignment – as her allocation. We denote by $\phi_i(\succeq)$ the allocation of agent i by rule ϕ in problem $\succeq$.

We will use the following concept of efficiency for random assignments due to Bogomolnaia and Moulin (2001).

Definition 1.

1. *An allocation x_i for agent i* **first-order stochastically** *dominates (sd-dominates) another allocation x'_i if, for each $o \in O$:*

$$\sum_{o':o' \succeq_i o} x_{io'} \geq \sum_{o':o' \succeq_i o} x'_{io'}.$$

In this case, we use the notation $x_i \succeq_i^{sd} x'_i$.

2. *An assignment x* **stochastically dominates** *another assignment x' if for every agent i, $x_i \succeq_i^{sd} x'_i$, and $x \neq x'$.*

3. *An assignment is* **sd-efficient** *if no other assignment stochastically dominates it.*

The standard notion of fairness requires that each agent should find her allocation at least as desirable as anyone else's allocation.

Definition 2. *Given a problem $\succeq$, an assignment x is* **sd-envy-free** *for $\succeq$ if for all $i, j \in I$, we have $x_i \succeq_i^{sd} x_j$.*

If instead we require that there is no other agent's allocation that sd-dominates the agent's allocation, we have a weaker notion, called **weak sd-envy-freeness**.

The following requirement is that assignments should be independent of the agents' names.

Definition 3. *Given a problem $\succeq = (\succeq_i)_{i \in I}$ and a permutation $\pi : N \to N$, a rule ϕ satisfies* **anonymity** *if for each $i \in I$:*

$$\phi_i((\succeq_{\pi(i)})_{i \in I}) = \phi_{\pi(i)}((\succeq_i)_{i \in I}).$$

Anonymity implies a weaker notion of fairness called **equal treatment of equals**, according to which agents with the same preferences should obtain the same allocation.

Finally, we consider the requirement of immunity to misrepresentations of individual preferences.

Definition 4. *A rule ϕ is* **sd-strategyproof** *if at any preference profile $\succeq$ no agent benefits by misreporting her preferences: for each $\succeq$, for each $i \in I$, and for each $\succeq'_i$:*

$$\phi_i(\succeq) \quad \succeq_i^{sd} \quad \phi_i(\succeq'_i, \succeq_{-i}).$$

As with sd-envy freeness, if in the last definition we require that there be no manipulation such that its outcome sd-dominates the allocation the individual gets from truth-telling, we get a weaker notion, called **weak sd-strategyproofness.** It is weaker because it allows the allocation the agent gets when she manipulates to not be comparable to her allocation under truth-telling.

We will compare our proposal, the Rawlsian rule, with the PS rule due to Bogomolnaia and Moulin (2001) which is defined as follows. Given a problem, think of each object as an infinitely divisible good with unit supply. **Step 1**: All agents start by consuming probabilities of receiving their most preferred object at the same unit speed. Proceed to the next step when an object is completely exhausted. **Step** $s \geq 2$: All agents consume probabilities of receiving their remaining most preferred object at the same speed. Proceed to the next step when an object is completely exhausted. The procedure terminates when each agent has eaten exactly one total unit of objects (i.e., at time one). The allocation of an agent i is given by the amount of each object she has eaten until the algorithm terminates.

4 Rawlsian assignments

In this section, we define our main concept of Rawlsian assignments. Given an assignment x, we denote by $b_i^x(k)$ the total probability with which agent i gets objects with a rank between n to k in her preferences. That is:

$$b_i^x(k) = \sum_{o \in O} \mathbb{1}\{r_{io} \geq k\} x_{io}.$$

In particular, $b_i^x(n)$ is the probability with which agent i receives her least preferred object. By definition, $b_i^x(1) = 1$. We denote by b_i^x the vector of the cumulative probability from the least to the most preferred object: $b_i^x = (b_i^x(n), b_i^x(n-1), \ldots, b_i^x(1) = 1)$.

Note that the previous definition is different from the standard concept of stochastic dominance where the cumulative probability is computed from the most to the least preferred object, that is:

$$\sum_{o \in O} \mathbb{1}\{r_{io} = 1\} x_{io}, \sum_{o \in O} \mathbb{1}\{r_{io} \leq 2\} x_{io}, \ldots, \sum_{o \in O} \mathbb{1}\{r_{io} \leq n\} x_{io} = 1.$$

Given an assignment x, and the vectors $(b_i^x)_{i \in I}$, we define the vector $B^x \in [0,1]^{n^2}$ as follows.

1. The first elements $(B_1^x, \ldots, B_n^x)$ are the elements $(b_1^x(n), \ldots, b_n^x(n))$ listed in a non-increasing order.
2. Elements $(B_{n+1}^x, \ldots, B_{2n}^x)$ are the elements $(b_1^x(n-1), \ldots, b_n^x(n-1))$ listed in a non-increasing order.
3. In general, elements $(B_{(k-1)n+1}^x, \ldots, B_{kn}^x)$ for $k = 1, \ldots, n$, are the elements $(b_1^x(n-k+1), \ldots, b_n^x(n-k+1))$ listed in a non-increasing order.

Vector B^x describes the cumulative distribution of probabilities induced by assignment x, from the least preferred object of each agent, to her most preferred object. The first n entries of the vector are the probabilities with which each agent receives her least preferred object. And, in particular, the first entry is the highest of the previous probabilities. Note that this definition is different from the one used in the leximin order. The coordinates of B^x are not listed in a non-increasing order but in blocks of n elements, and within each block, the elements are listed in a non-increasing order. This is different from the definition of PS rule proposed by Bogomolnaia (2015, 2018) who orders all the elements of the vector.

The following example illustrates the definition.

Example 1. *Let $I = \{1, 2, 3\}$, $O = \{a, b, c\}$, and preferences as follows:*

$$\begin{array}{c|ccc} \succeq_1 & a & b & c \\ \succeq_2 & a & b & c \\ \succeq_3 & b & c & a \end{array}$$

Consider the assignment:

$$x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then: $b_1^x = b_2^x = \left(0, \frac{1}{2}, 1\right)$, $b_3^x = (0, 1, 1)$, and $B^x = \left(0, 0, 0, 1, \frac{1}{2}, \frac{1}{2}, 1, 1, 1\right)$. $\square$

To compare two assignments, x and y, we first compare the highest probability with which an agent receives her least preferred object, that is, the first entry of vectors B^x and B^y. If the probability is the same under both assignments, we compare the second highest probability with which an agent receives her least preferred object. If all probabilities associated with the least preferred object of each agent are equal, we conduct the same comparison for the probabilities with which agents receive the least and second-least preferred objects. If at some point, an entry of B^x is lower than the corresponding entry of B^y, we say that x Rawlsian-dominates y. Formally, we compare the vectors B^x and B^y lexicographically. This is the idea of the following definition.

Definition 5. *Given two assignments x and y, x* **Rawlsian-dominates** *y (x R-dominates y) if there is $j \in \{1, \ldots, n^2\}$ such that $B_j^x < B_j^y$, and for all $i < j$, $B_i^x = B_i^y$.*

The following is the key concept of our analysis.

Definition 6. *An assignment x is* **Rawlsian** *if it is not Rawlsian-dominated (R-dominated) by any other assignment.*

Example 2 shows a Rawlsian assignment for the previous example.

Example 2. *In the problem of Example 1, consider the assignment:*

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

to which is associated the vector $B^y = (1, 0, 0, 1, 0, 0, 1, 1, 1)$. Clearly, x R-dominates y. Moreover, we claim that x is a Rawlsian assignment. First, note that if another assignment x is such that $x_{3c} < 1$, then we should have $x_{1c} > 0$ or $x_{2c} > 0$. This new assignment would be R-dominated by x. So, if an assignment, say z, R-dominates x, it should satisfy $z_{1b} < \frac{1}{2}$ or $z_{2b} < \frac{1}{2}$. In the first case, $z_{2b} > \frac{1}{2}$ should hold, and z would be R-dominated by x. In the second case, $z_{1b} > \frac{1}{2}$ should hold, and z would be R-dominated by x. As a result, no assignment R-dominates x. □

To better understand the intuition behind the Rawlsian rule, we will find a Rawlsian assignment for some particular preference profiles. First, if all agents' top choices are different, a Rawlsian assignment assigns each agent their first choice with probability one. Alternatively, if all agents' preferences are the same, each agent is assigned each object with probability $\frac{1}{n}$. Lastly, an agent is assigned her least preferred object if, and only if, all agents have the same least preferred object.

Given an assignment x, we defined the vector B^x by considering, in the first place, the probabilities associated with the least preferred object of each agent, that is, the object of rank n. Next, we consider the two least preferred objects, the objects of rank $n-1$ or higher. And so on, and so forth. The order $(n, n-1, \ldots, 2)$ comes from our definition of the worst-off agents.[1] However, others orders are also possible. For example, we might consider the other "extreme" of the Rawlsian assignment: first consider the probabilities with which each agent is assigned to her $(n-1)$ least preferred objects (rank 2 or higher), then her $(n-2)$ least preferred objects (rank 3 or higher), etc. The associated order would be $(2, 3, \ldots, n)$.

In Appendix 10.2, we consider a family of assignments where each member is defined by considering a different order with which the elements of the vector $(b_i^x)_{i\in I}$ are listed. For example, the Rawlsian assignment corresponds to the order $(n, n-1, \ldots, 2)$. We show that many of the results in Sections 5 remain to hold, including uniqueness, sd-efficiency, and anonymity of each of the rules in this family. Additionally, a modified version of the algorithm in Section 6 can be used to compute the outcomes for each of the rules in this family. We also show in Appendix 10.3 that the other "extreme" of the Rawlsian assignment in this family, the one associated with the order $(2, 3, \ldots, n)$, differs from the fractional Boston assignment introduced by Chen, Harless, and Jiao (2023), despite the similarities between both assignments.

5 Results

Every problem admits a Rawlsian assignment. In principle, there might exist multiple Rawlsian assignments but, in fact, this is never the case. Suppose otherwise, and let x and y be two Rawlsian assignments. Then, the assignment $\frac{1}{2}x + \frac{1}{2}y$ R-dominates x and y. Thus, as the following proposition states, every problem admits a unique Rawlsian assignment. It is worth

[1]Because the probability with which each agent is assigned to her n least preferred objects is always one, we do not include 1 in this order.

noting that randomization is key for the uniqueness. If we restrict the analysis to deterministic assignments, there are problems with multiple Rawlsian assignments.

Proposition 1. *Each problem has a unique Rawlsian assignment.*

A minimum requirement of fairness is anonymity: agents' allocations cannot depend on their names, that is, if agents' names are permuted, their allocations should be permuted in the same way. As we state in the next proposition, the Rawlsian rule satisfies this property.

Proposition 2. *The Rawlsian rule satisfies anonymity.*

Example 3 shows that a Rawlsian assignment may not be sd-envy-free. Moreover, the same example shows that it may not even be weakly sd-envy-free. In fact, there is no relation between Rawlsian assignments and sd-envy-free assignments. In Example 1, the outcome of PS rule is sd-envy-free but not Rawlsian.

Example 3. *Let $I = \{1, 2, 3\}$, $O = \{a, b, c\}$, and preferences as follows:*

$$\begin{array}{c|ccc} \succeq_1 & a & b & c \\ \succeq_2 & b & a & c \\ \succeq_3 & b & c & a \end{array}$$

The Rawlsian assignment is:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that agent 3's allocation is first-order stochastically dominated by agent 2's allocation (considering agent 3's preferences). □

Duddy (2022) introduces a notion of egalitarian assignments. An assignment y is **inegalitarian** if there exists another assignment x and an agent j such that the allocations of all the agents at x sd-dominate the allocation of j at y. That is, there is another assignment at which each agent prefers her allocation to agent j's allocation at y. An egalitarian assignment is an assignment that is not inegalitarian. We formally define this concept in Appendix 10.5, and show that it is independent to the concept of a Rawlsian assignment.

5.1 Efficiency

We now analyze the efficiency of Rawlsian assignments. In contrast to many economic environments where there is a tension between egalitarianism and efficiency, these concepts are compatible in our framework. Given two assignments, x and y, we define the following order to compare the associated vectors B^x and B^y.

Definition 7. *Let x and y be two assignments. We say that $B^x \lhd B^y$ if $b_i^x(k) \leq b_i^y(k)$ for every $i, k = 1, \ldots, n$, and $b_i^x(k) < b_i^y(k)$ for some $i, k = 1, \ldots, n$.*

That is, every entry of the vector B^x is lower than or equal to the corresponding entry of the vector B^y. As we show in the next lemma, one can characterize stochastic dominance using the relation $\lhd$.

Lemma 1. *Let x and y be two assignments. Then, x stochastically dominates y if, and only if, $B^x \lhd B^y$.*

Proof. By definition x stochastically dominates y if, $x \neq y$ and for every $i = 1, \ldots, n$:

$$1 - b_i^x(k) \geq 1 - b_i^y(k) \iff b_i^x(k) \leq b_i^y(k) \text{ for every } k = 1, \ldots, n.$$

Moreover, as $x \neq y$, one of the previous inequalities should be strict. Thus, x stochastically dominates y if, and only if, $B^x \lhd B^y$. □

Consider an assignment y which is not sd-efficient. This implies that there exists another assignment x that sd-dominates y. Then, every element of B^x is lower than or equal to the corresponding element of B^y. But then, x Rawlsian dominates y, and therefore, y is not the Rawlsian assignment. This proves the following proposition.

Proposition 3. *A Rawlsian assignment is sd-efficient.*

It is easy to see that the converse of the proposition is not true: there are sd-efficient assignments that are not Rawlsian. The following example shows that the PS rule, which is sd-efficient, is different from the Rawlsian rule.

Example 4. *Consider the problem of Example 1. The outcome of the PS rule (which coincides in this case with the random serial dictatorship) is:*

$$y^{PS} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

The outcome of PS rule is sd-efficient and for this problem is different from the Rawlsian assignment. □

The Birkhoff-von Neumann theorem states that any assignment can be represented as a lottery over the set of deterministic assignments. The Rawlsian assignment is sd-efficient; thus, **every** such representation will only contain ex-post efficient deterministic assignments (Aziz, Mackenzie, Xia, and Ye, 2015). Define the **maximum rank** of a deterministic assignment as follows. For each agent, consider the rank in the agent's preferences of her assigned object, and then take the maximum. Then, we have the following corollary.

Corollary 1. *Every representation of the Rawlsian assignment as a lottery over the set of deterministic assignments uses only ex-post efficient assignments that minimize the maximum rank.*

There is another concept of efficiency, called "rank efficiency", recently proposed by Featherstone (2020). In Appendix 10.4, we give its formal definition and show that there is no relation between Rawlsian and rank efficient assignments.

5.2 Strategyproofness

We have shown that the Rawlsian rule is sd-efficient and satisfies equal treatment of equals (as a consequence of being anonymous). Bogomolnaia and Moulin (2001) showed that, when there are least four agents, no rule is sd-efficient, strategyproof, and satisfies equal treatment of equals. Thus, we have the following corollary.

Corollary 2. *The Rawlsian rule is not sd-strategyproof.*

Our next example illustrates possible manipulations of the Rawlsian rule.

Example 5. *Let $I = \{1, 2, 3\}$, $O = \{a, b, c\}$, and*

$$\begin{array}{c|ccc} \succeq_1 & a & b & c \\ \succeq_2 & b & c & a \\ \succeq_3 & b & c & a \end{array}$$

The Rawlsian assignment of the problem $(\succeq_1, \succeq_2, \succeq_3)$ is:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Suppose agent 2 reports the preferences $\succeq_2' = (b, a, c)$. Then, the Rawlsian assignment of the problem $(\succeq_1, \succeq_2', \succeq_3)$ is:

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Agent 2 prefers (according to $\succeq_2$) her allocation under y to her allocation under x. $\square$

As we mentioned before, a weaker notion of sd-strategyproofness requires that the outcome under an individual manipulation never dominates the allocation obtained when the agent reveals her true preferences. The previous example shows that the Rawlsian rule is not weakly sd-strategyproof. Moreover, this example also shows that it is obviously manipulable according to an extension to the probabilistic setting of a definition due to Troyan and Morrill (2020). See

Appendix 10.6 for details. Lastly, in Appendix 10.7 we also show that the Rawlsian rule satisfies the lower invariance axiom in the decomposition of strategyproofness by Mennle and Seuken (2021), but violates the axioms of upper invariance and swap monotonicity.

5.3 The Rawlsian assignment and social welfare functions

We conclude this section by discussing the relation between the Rawlsian assignment and agents' cardinal utilities. McLennan (2002) characterizes sd-efficiency in terms of the existence of cardinal utilities that maximize agents' total welfare. He shows that an assignment is sd-efficient if, and only if, there exists a profile of cardinal utilities consistent with the agents' ordinal preferences for which the assignment maximizes the utilitarian welfare (see also Manea (2008)).

One may wonder if an equivalent result holds in terms of the Rawlsian assignment. That is: is it possible to characterize the Rawlsian assignment by the existence of cardinal utilities that maximize the expected utility of the worst-off agents? It turns out that if an assignment is sd-efficient, there exists such a cardinal representation. That is, if x is sd-efficient, there exist cardinal utilities (consistent with the ordinal preferences) such that x verifies

$$x \in \arg\max_{x' \in X} \{\min_{i \in I} U_i(x')\},$$

where $U_i(x') = \sum_{o \in O} x'_{io} u_{io}$, and u_{io} is agent i's cardinal utility for object o. The Rawlsian assignment is sd-efficient, so the result implies that for each problem $\succeq$ there exist cardinal utilities for which the Rawlsian assignment maximizes the expected utility of the worst-off agent among all possible assignments (see Appendix 10.8 for a proof).

What can we say about the converse of the last result? Consider an assignment that maximizes the welfare of the worst-off agent for some cardinal representation of the ordinal preferences. Is it a Rawlsian assignment? The next proposition states that this is not the case. Moreover, Proposition 4 implies a stronger result: no rule treats equals equally and maximizes the utility of the worst-off agent for all cardinal utilities consistent with the ordinal preferences. In particular, we include an example where for different cardinal utilities (each consistent with the given ordinal preferences), a different assignment maximizes the utility of the worst-off agent. And for some of these cardinal utilities, the resulting probabilistic assignment is not equal to the Rawlsian assignment.

Proposition 4. *Given a problem $\succeq$, no rule treats equals equally and maximizes the utility of the worst-off agent for all cardinal utilities consistent with ordinal preferences $\succeq$.*

6 Computing the Rawlsian assignment

We now define an algorithm to compute the Rawlsian assignment in polynomial time. The proposed algorithm is an extension of the algorithm by Airiau, Aziz, Caragiannis, Kruger, Lang, and Peters (2023), which computes a leximin distribution of a divisible object, to our setting with multiple objects. The pseudo-code is shown in Algorithm 1 and an implementation of the code is available online (https://github.com/DemeulemeesterT/Rawlsian-assignments).

Intuitively, the algorithm finds the Rawlsian assignment x by lexicographically minimizing the corresponding vector $B^x \in [0,1]^{n^2}$ as defined in Section 4. To do so, it first finds the lowest probability with which any agent can be assigned her n-th choice, i.e., $\min_{i \in I} b_i^x(n)$, by solving a first linear program. Denote by b^* the optimal value of this program, which we can solve efficiently using solvers like CPLEX or Gurobi.

Now that we know that all agents can be assigned their n-th ranked object with a probability of at most b^*, we find the agents for whom this probability is exactly b^*. There must be at least one agent for which this is the case, otherwise the optimal solution of the first linear program would have been strictly lower than b^*. If the optimal solution ϵ^* of the second linear program is equal to zero for some agent i', it is not possible to assign that agent her n-th choice with a probability strictly lower than b^* while ensuring that all other agents are assigned their n-th choice with a probability of at most b^*. Accordingly, we add an additional constraint to the remaining linear programs that are solved in Algorithm 1, namely that agent i' be assigned her n-th choice with a probability equal to $b_{i'}^x(n) = b^*$. Moreover, we add agent i' to I_n, which is the set of agents for which the probability of being assigned their n-th choice is already fixed by the algorithm.

Next, we find the lowest probability with which any of the remaining agents in $I \setminus I_n$ is assigned their n-th choice. As before, we determine the remaining agents in $I \setminus I_n$ for whom this is the exact probability of being assigned their n-th choice. We repeatedly solve the first and second linear programs until $I_n = I$.

Finally, we repeat the above procedure for each of the n preferences. In general, I_k is the set of agents for which the algorithm has already found the exact probability with which they are assigned an object of rank $k, \ldots, n$. It can be checked that Algorithm 1 needs to solve at most $n^2 \cdot \frac{n+1}{2}$ linear programs to find the Rawlsian assignment. Note that by changing the order in which all agent-preference pairs are considered, a similar algorithm can be used to compute the probabilistic assignments of alternative rules, such as the positive equality rule by Duddy (2022), or any of the generalized rules defined in Appendix 10.2 that compare assignments based on an alternative ordering of the elements of the vector $(b_i^x)_i$.

Algorithm 1 Computing the Rawlsian assignment

$I_k \leftarrow \varnothing$ for $k \in \{1, \ldots, n\}$.
$b_i^x(k)$ is fixed once $i \in I$ is added to I_k.

for $k \in \{n, \ldots, 1\}$ **do**
  while $I_k \neq I$ **do**
    Find the minimum value of b^* such that there exists an assignment $x \in [0,1]^{n^2}$ satisfying

$$\begin{aligned} \sum_{o \in O} x_{io} &= 1 && \forall\, i \in I \\ \sum_{i \in I} x_{io} &= 1 && \forall\, o \in O \\ \sum_{o \in O} \mathbb{1}\{r_{io} \geq k\} x_{io} &\leq b^* && \forall\, i \in I \setminus I_k \\ \sum_{o \in O} \mathbb{1}\{r_{io} \geq t\} x_{io} &= b_i^x(t) && \forall\, i \in I_t : t \geq k \end{aligned}$$

    if $b^* = 0$ **then** $I_k \leftarrow I$
    else
      for $i' \in I \setminus I_k$ **do**
        Find the maximum value of ϵ^* such that there exists an assignment $x \in [0,1]^{n^2}$ satisfying

$$\begin{aligned} \sum_{o \in O} x_{io} &= 1 && \forall\, i \in I \\ \sum_{i \in I} x_{io} &= 1 && \forall\, o \in O \\ \sum_{o \in O} \mathbb{1}\{r_{i'o} \geq k\} x_{i'o} &\leq b^* - \epsilon^* && \\ \sum_{o \in O} \mathbb{1}\{r_{io} \geq k\} x_{io} &\leq b^* && \forall\, i \in I \setminus I_k \\ \sum_{o \in O} \mathbb{1}\{r_{io} \geq t\} x_{io} &= b_i^x(t) && \forall\, i \in I_t : t \geq k \end{aligned}$$

        if $\epsilon^* = 0$ **then** add i' to I_k, and set $b_{i'}^x(k) = b^*$.
        end if
      end for
    end if
  end while
end for
return The solution $x \in [0,1]^{n^2}$ from the last solved LP.

7 Empirical Application

Our main motivation is the assignment of apartments in housing cooperatives. A housing cooperative is formed by a group of families who join to construct a building. Once it is finished, the apartments are to be distributed. Prices are not used, so the situation fits the assignment problem previously described.

Before the rule currently in use, cooperatives assigned apartments randomly. Preferences were not considered and the assignment was sampled randomly from the set of all assignments. The assignment was in general not efficient, so it was decided to change the rule. A group of researchers from the Engineering School of the University of Uruguay, proposed a new rule, called the **MTAV**, which was finally adopted. More information about the current rule can be found in Prino, Sánchez, and Cancela (2016) and Paleo (2021).

One of the main concerns of the families that participate in the cooperatives is the *distribution* of the final assignment. They do not want *inegalitarian* distributions in terms of the ranking of the assigned apartment in each family's preferences. They want to avoid a situation where, for example, a family gets their most preferred unit, while others get apartments ranked very low in their preferences. The MTAV explicitly addresses this concern (we define it in Appendix 10.9).

In this section, we use the data of families' preferences from 24 cooperatives to compare the outcomes of the Rawlsian, PS and MTAV rules. In practice, the assignment is organized according to the number of rooms in the apartments. Therefore, all the apartments we consider as part of a cooperative have the same number of rooms. The sizes of the cooperatives range from 4 to 42 families (with an average size of 17). As a proxy for the correlation of preferences, we consider the cumulative number of different apartments ranked in the first, second, third, and fourth position in the preferences. The result shows that preferences are not highly correlated. In Appendix 1 we present detailed information for each cooperative.

It is worth noting that there are only two cooperatives, C_4 and C_{10}, where the Rawlsian and the PS rules coincide. These are the smallest cooperatives (each consisting of 4 families). Additionally to these 8 families, there is only one family which receives the same allocation under the two rules. Thus, overall, 9 families out of 408 receive the same lottery over apartments.

We should mention that the MTAV is not strategyproof (Paleo, 2021). Nonetheless, we take preferences submitted by the families at face value. We are not aware of manipulations by the families, and in general given the information held by the families, it is very difficult to manipulate the rule profitably.

7.1 Comparison with PS rule: Maximum rank

The Rawlsian rule is sd-efficient and it is designed to improve the welfare of the worse-off families. A first way to measure this improvement is what we call the maximum rank. Given an assignment, we consider for each agent the rank of the least preferred object received with positive probability. Then, we take the maximum rank among all families. In Table 1 we look at the maximum rank of the Rawlsian and PS rules. In contrast to what might be expected (based on the correlation of preferences), in all but two cooperatives, the PS rule assigns at least one family their least preferred apartment with positive probability. Under the Rawlsian rule, this happens only in three cooperatives (those with a small number of families). For the rest of the cooperatives, the average maximum rank of the Rawlsian rule as a percentage of the length of families' preferences is 48%.

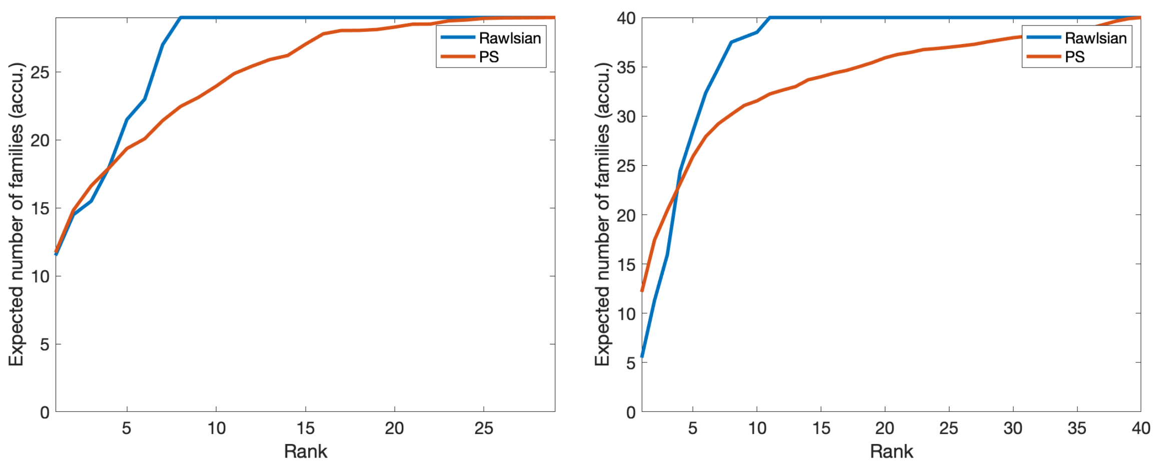
Table 1: Size and maximum rank of each cooperative for the Rawlsian and PS assignment.

Coop.	Size	Max. Rawls	Max. Rawls (%)	Max. PS
C_1	26	13	50	26
C_2	18	12	67	18
C_3	4	2	50	4
C_4	4	3	75	3
C_5	28	8	29	28
C_6	8	3	38	8
C_7	29	8	28	29
C_8	12	7	58	12
C_9	15	6	40	14
C_{10}	4	4	100	4
C_{11}	11	5	45	11
C_{12}	16	6	38	16
C_{13}	39	14	36	39
C_{14}	42	33	79	42
C_{15}	14	9	64	14
C_{16}	6	6	100	6
C_{17}	9	3	33	9
C_{18}	15	8	53	15
C_{19}	9	9	100	9
C_{20}	20	10	50	20
C_{21}	24	7	29	24
C_{22}	7	2	29	7
C_{23}	40	11	28	40
C_{24}	8	7	88	8

Notes: Coop. stands for cooperative, each denoted as C_i for $i = 1, \ldots, 24$. Size is the number of families in each cooperative. For Max. Rawls (Max. PS) we compute the rank of the least preferred object assigned with positive probability by the Rawlsian (PS) rule for each family, and then we take the maximum among all families. Max. Rawls (%) expresses Max. Rawls as a percentage of the length of families' preferences (or, equivalently, the size of the cooperative).

The previous analysis shows that the Rawlsian assignment assigns fewer families their least preferred apartments. Now we look at the intensive margin, that is, the probabilities with which families are assigned their least preferred objects. It could be that even when the PS rule assigns families apartments that are ranked very low, this occurs with very small probability. To investigate this, we define the expected number of families that are assigned apartments ranked in position $k \in \{1, \ldots, n\}$ as the sum of the probabilities with which each family is assigned the apartment they rank in position k. Tables 5 - 10 in the Online Appendix present the results for all the cooperatives considering the Rawlsian and PS rules. Not only is the maximum rank higher under PS than under the Rawlsian assignment for each cooperative, but also the cumulative probability of being assigned their least preferred apartments is substantially higher. For example, for cooperative C_{13} the Rawlsian rule assigns all families apartments ranked 14th or better (out of 39 apartments), while the PS rule assigns (in expectation) 8 families apartments ranked 15th or worse.

The general picture regarding the expected number of families that are assigned apartments with rank k is as follows. The Rawlsian rule assigns a lower number of families their least preferred apartments compared to the PS rule. But, at the same time, it also assigns a lower number of families their top choices, and especially their first choice. As an illustration we include in Figure 2 the distribution of the expected number of families that are assigned apartments with rank k by each rule for two cooperatives (the reader can find the cumulative distribution for all the cooperatives and the Rawlsian, PS and MTAV rules here).



(a) Cooperative C_7

(b) Cooperative C_{23}

Figure 2: Cumulative distribution function of the expected number of families that are assigned apartments with rank k by the Rawlsian and PS rules.

7.2 Comparison with the PS rule: individual preferences over assignments

Both the Rawlsian and the PS rules are sd-efficient. Therefore, it is never the case that all families prefer the assignment under one assignment over the other. In this section we compare, for each cooperative, the number of families that prefer the assignment of one rule over the other. We say that a family prefers the Rawlsian (PS) assignment to the PS (Rawlsian) assignment if the lottery the families receives in the first assignment sd-dominates the lottery in the second assignment. Obviously, there are families for whom neither of the assignment dominates the other. Table 2 presents the results. For every cooperative, more families prefer their Rawlsian assignment to their PS assignment. Moreover, the average percentage of families that prefer their Rawlsian (PS) assignment over the PS (Rawlsian) assignment is 35% (9%).[2]

[2]One may wonder if the Rawlsian allocation is always more popular that the PS allocation (that is, if the number of agents who prefer the former is always higher than the number of agents who prefer the latter). The

Table 2: Number of families who prefer the assignment under the Rawlsian (PS) over the PS (Rawlsian) rule.

Coop.	Size	Prefer Rawls	Prefer PS
C_1	26	8	1
C_2	18	6	1
C_3	4	3	1
C_4	4	0	0
C_5	28	10	1
C_6	8	4	1
C_7	29	12	2
C_8	12	6	1
C_9	15	4	1
C_{10}	4	0	0
C_{11}	11	5	2
C_{12}	16	8	1
C_{13}	39	9	1
C_{14}	42	7	1
C_{15}	14	3	1
C_{16}	6	3	2
C_{17}	9	4	1
C_{18}	15	5	1
C_{19}	9	5	1
C_{20}	20	2	1
C_{21}	24	9	1
C_{22}	7	4	1
C_{23}	40	6	2
C_{24}	8	2	0

7.3 Comparison with the PS rule: Sd-envy-freeness.

In this section we turn to the analysis of sd-envy-freeness. The PS assignment is sd-envy-free, so no family envies another family under its outcome. As observed before, the Rawlsian rule is not sd-envy-fee, so some families may experience envy.

We show in Table 3, for each cooperative, the number of families (and the percentage over all the families) whose allocation does not sd-dominate the allocation of some other family. Also, among those families that experience envy, we show the average number of families that are sd-envied. There is only one cooperative where no family experiences envy. Among the rest, the percentage of families with sd-envy ranges from 22% to 94%, with an average of 58%. If we consider weak sd-envy, that is, a family has weak sd-envy over another family if the allocation of this last family sd-dominates the allocation of the first one, the percentage of families with weak sd-envy ranges from 11% to 67%, with an average of 36%.

7.4 Comparison with MTAV

Tables 5 - 10 in the Appendix include the expected number of families that are assigned apartments with rank k by each of the three rules. By construction, the maximum rank of the Rawlsian rule and MTAV always coincide. However, it is interesting to note that there are cases where the Rawlsian rule assigns fewer families their least preferred apartment (among those that are received with positive probability). For example, consider C_1: under the Rawlsian rule only one family receives their apartment of maximum rank (which is 13, out of 26 apartments), while under MTAV two families receive their 13th choice. The same is true for cooperatives C_2, C_5, C_9, C_{13}, C_{15}, C_{17}, C_{21}, and C_{23}. In general, the outcome of the MTAV is located between the two other rules. Indeed, the Rawlsian rule outperforms the MTAV under the Rawlsian criterion, but not in terms of the expected number of families assigned their top choice. The PS rule outperforms the MTAV rule in terms of the expected number of families assigned their top choice, but not under the Rawlsian criterion.

8 Analysis for large markets

In the empirical application, we show that the average maximum rank of the Rawlsian assignment (excluding those cooperatives with less than 10 families) is around 46% of the market size (Section 7.1). In this section, we build on this finding by analyzing the maximum rank of the Rawlsian rule in large markets. Specifically, we consider markets of size n, where agents' preferences are drawn i.i.d. from a uniform distribution, and study the limit as n tends to infinity. Although the average maximum rank of the Rawlsian rule tends to infinity, we show that it grows

example in Remark 2 (Appendix 10.4) shows that this is not the case. Indeed, in this problem, two agents prefer the PS assignment, one agent prefers the Rawlsian assignment, one agent obtains the same allocation, and one agent does not prefer either assignment based on stochastic dominance.

Table 3: Sd-envy in the Rawlsian assignment.

Coop.	Size	Envy Fam.	Envy Fam. (%)	Avg. Envied Fam.
C_1	26	20	77	6
C_2	18	17	94	6
C_3	4	1	25	2
C_4	4	0	0	0
C_5	28	18	64	5
C_6	8	4	50	1
C_7	29	18	62	5
C_8	12	6	50	3
C_9	15	10	67	4
C_{10}	4	0	0	0
C_{11}	11	5	45	2
C_{12}	16	10	63	3
C_{13}	39	33	85	7
C_{14}	42	40	95	12
C_{15}	14	11	79	5
C_{16}	6	2	33	3
C_{17}	9	2	22	3
C_{18}	15	10	67	5
C_{19}	9	4	44	6
C_{20}	20	18	90	7
C_{21}	24	15	63	5
C_{22}	7	3	43	1
C_{23}	40	36	90	6
C_{24}	8	7	88	3

Notes: Envy Fam. is the number of families with sd-envy, Envy Fam. (%) is the percentage of families with sd-envy, and Avg. Envied Fam. is the average of the number of sd-envied families.

at a slow rate. In particular, the expected maximum rank for the Rawlsian rule is upper bounded by $\lfloor \ln(n) \rfloor$ plus a constant. For instance, in a market of size $n = 1000$, our result implies that the expected rank of the least preferred assigned object is at most 9.

Let $\mathcal{P}_n$ denote the set of all possible preference profiles in a market of size n. We define the random variable $\succeq \in \mathcal{P}_n$ that selects one of the preference profiles in $\mathcal{P}_n$ uniformly at random. Then, we denote the maximum rank of the assignment found by rule ϕ at $\succeq$ as:

$$r^{\phi}_{max}(\succeq) = \max_{i \in I} r_{i,\phi_i(\succeq)}.$$

The expected maximum rank of the rule ϕ is then defined as:

$$\mathbb{E}(r^{\phi}_{max}) = \sum_{\succeq \in \mathcal{P}_n} \frac{1}{|\mathcal{P}_n|} r^{\phi}_{max}(\succeq). \tag{1}$$

First, we show that the expected maximum rank of the Rawlsian assignment goes to infinity when the market grows large.

Proposition 5. $\lim_{n \to \infty} \mathbb{E}(r^{Rawls}_{max}) = +\infty$.

Next, we provide an upper bound on the growth rate of the Rawlsian rule's maximum rank.

Proposition 6. *Consider the Rawlsian rule. Then, when $n \to \infty$:*

$$\mathbb{E}_n(r^{Rawls}_{max}) \leq \lfloor \ln(n) \rfloor + \sum_{k=-1}^{+\infty} \left(1 - e^{-2e^{-k}}\right) \approx \lfloor \ln(n) \rfloor + 2.77026. \tag{2}$$

The proofs of both results are included in Appendices 10.10 and 10.11. We are not aware of any theoretical result on the maximum rank of the PS rule. Ortega and Klein (2023) showed that the rank-minimizing rule, which minimizes the expected rank of agents, has an expected maximum rank of $\log_2(n)$ in large random markets. Interestingly, our result implies that the expected maximum rank of the Rawlsian rule in random markets converges to a fraction of at most $\ln(2) \approx 0.69$ of the maximum rank of the rank-minimizing rule.

To show the tightness of the bound in Proposition 6, we have randomly sampled $1,000$ preference profiles for each market of size $n \in \{3, \ldots, 59\}$. The left panel of Figure 3 shows that the empirical analysis in Section 7.1 regarding the difference in the maximum rank between the Rawlsian and PS rules seems to hold in general. The probability with which the PS rule assigns an agent to their last choice with positive probability is close to one. As the size of the market grows and preferences are uniformly i.i.d., the difference between the maximum rank of the Rawlsian and the PS rules tends to infinity on average. Therefore, the egalitarian advantage of the Rawlsian rule over PS also holds in large markets.

The right panel of Figure 3 shows that the upper bound of Proposition 6 is reasonably close to the observed maximum rank of the Rawlsian rule. Interestingly, the true maximum rank of the Rawlsian rule seems to be approximately equal to $\ln(n) + 1$.

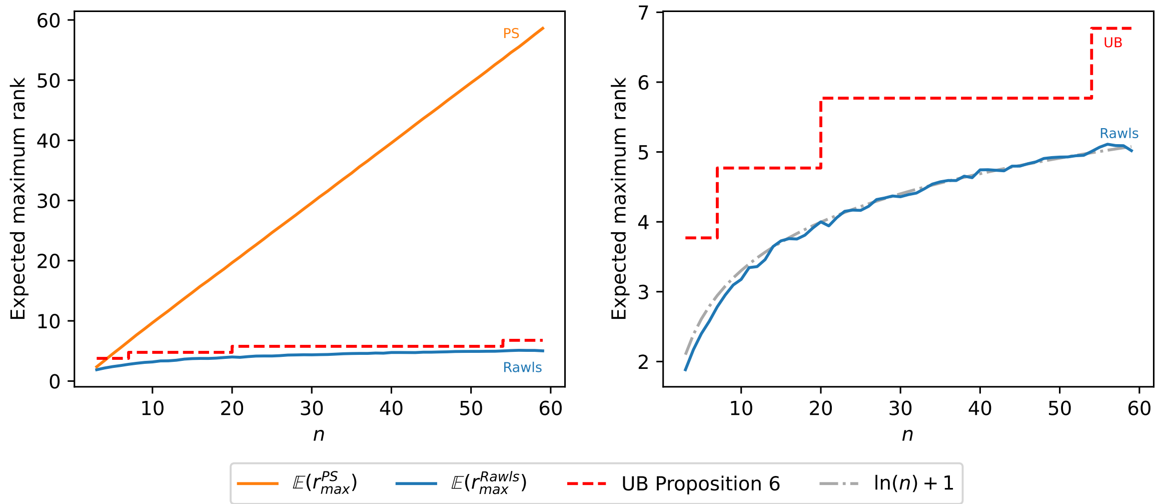


Figure 3: Simulation results maximum ranks Rawlsian and PS rules.

9 Concluding Remarks

We examined the allocation of indivisible goods to individuals when prices cannot be utilized. Our investigation draws inspiration from the context of housing cooperatives, where families express concerns about the fairness of the final assignment. Specifically, we aim to avoid assignments in which some families receive their top choices while others are assigned apartments ranked very low in their preferences. To address this, we introduce a concept called Rawlsian assignments, which prioritizes improving the allocations of individuals who are worst-off. We demonstrate that there always exists a unique Rawlsian assignment. Moreover, the Rawlsian rule is both sd-efficient and anonymous. Furthermore, we compare our proposed rule with the PS rule and the currently employed rule. Our findings reveal that the Rawlsian rule significantly outperforms the other two rules in terms of egalitarianism outcomes. However, as highlighted in Proposition 4, it is not the sole rule that maximizes the utility of the worst-off agent. Consequently, exploring alternative rules that prioritize egalitarianism represents an intriguing avenue for future research.

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10 Appendix

10.1 Proofs

10.1.1 Proof of Proposition 1.

Proof. **Existence** The set of assignments X is a compact set of $[0,1]^{n^2}$. For each $x \in X$ we construct the vector $B^x = (B^x_1, \dots, B^x_n, B^x_{n+1}, \dots, B^x_{2n}, \dots, B^x_{n^2})$. Consider the function $\pi_1 : X \to \mathbb{R}$ such that $\pi_1(x) = B^x_1$. Clearly, π_1 is a continuous function (it is a projection). Then, the problem $\min_{x \in X} \pi_1(x)$ has a solution. Let $S_1 \subset X$ be set of all solution to the minimization problem.

Note that S_1 is compact. It is bounded because X is bounded. It is also closed: take a convergent sequence $\{x^k\}_k \subset S_1$. All the elements x^k have the same $B^{x^k}_1$, so the limit of the sequence must have the same $B^{x^k}_1$ as well. Then, the limit is an element of S_1.

Consider the function $\pi_2 : S_1 \to \mathbb{R}$ such that $\pi_2(x) = B^x_2$. Clearly, π_2 is a continuous function. Then, the problem $\min_{x \in S_1} \pi_2(x)$ has a solution. Let $S_2 \subset X$ be set of all the solutions to the minimization problem.

We continue in the same way for all the elements of the vector B^x. At the end, we will have a nonempty set S_{n^2}.[3] The assignments in this set are Rawlsian. Indeed, suppose this is not the case, and consider $x \in S_{n^2}$ which is Rawlsian-dominated by another assignment y. This means that there exists an index $j \in \{1, \dots, n^2\}$ such that $B^x_j > B^y_j$, and for all $i < j$, $B^x_i = B^y_i$. Then, $y \in S_{j-1}$, and $\pi_j(y) < \pi_j(x)$. This implies that $x \notin S_j \Rightarrow x \notin S_{n^2}$, which is a contradiction.

Uniqueness Suppose there are two Rawlsian assignments, x and y. Both are associated with the same vector B. Given an assignment z, consider a matrix P^z where agents are represented in the rows, and in column k we include the probability with which each agent receives the object ranked in position k.[4] Starting from the last column, consider the first column where P^x and P^y differ (there is such a column as x and y are different assignments). Assume that this is the case for column $n - c$.

Note that the probabilities of column $n - c$ in each matrix are the same, but distributed differently. Until column $n - c$, the two matrices are the same, so the same agents get the same probabilities for the corresponding objects. This implies that all agents receive the same probability for objects ranked $n - (c - 1), \dots, n$ under x and y. Therefore, they also receive the same probabilities under assignment $\frac{1}{2}(x + y)$.

Consider the largest element of column $n - c$ of each of the matrices P^x, P^y, and $P^{\frac{1}{2}(x+y)}$. If the largest element of this column in P^x and P^y corresponds to the same agent, then the probability with which this agent receives the object ranked in position $n - c$ coincides in x, y, and $\frac{1}{2}(x + y)$. If it corresponds to a different agent, then either the largest element of column $n - c$ in P^x or the largest element of column $n - c$ in P^y, is larger than the largest element of

[3]Note that there will be a unique vector B^x that minimizes the problems. But, in principle, we could have many assignment associated with the same the vector B^x.

[4]Matrix P^z is created by reordering each row of z based on the agents' preferences.

$P^{\frac{1}{2}(x+y)}$. In the first case, $\frac{1}{2}(x+y)$ R-dominates x, and in the second case $\frac{1}{2}(x+y)$ R-dominates y. But this contradicts the fact that x and y are Rawlsian assignments. As the assignments x and y differ in at least one entry of column $n-c$, the assignment $\frac{1}{2}(x+y)$ R-dominates x or y. $\square$

10.1.2 Proof of Proposition 2.

Proof. Consider two problems $(\succeq_i)$ and $(\succeq_{\pi(i)})$, and let $(x_i)_i$ and $(x_i')_i$ be the Rawlsian assignments in each of these problems. We need to show that $x_i' = x_{\pi^{-1}(i)}$ for every $i \in \{1, \dots, n\}$. Suppose this is not the case, and consider the assignments $(x_{\pi^{-1}(i)})_i$ and $(x_i')_i$, and the first entry where vectors $B^{(x_{\pi^{-1}(i)})_i}$ and $B^{(x_i')_i}$ differ. Because $(x_i')_i$ is the Rawlsian assignment of $(\succeq_{\pi(i)})_i$, this entry in $B^{(x_i')_i}$ is smaller than in the vector $B^{(x_{\pi(i)})_i}$. Note that the definition of the vector B^x of each assignment x, does not look at the identities of the agents, thus: $B^{(x_i)_i} = B^{(x_{\pi(i)})_i}$. But then $(x_i')_i$ R-dominates $(x_i)_i$ in problem $(\succeq_{\pi(i)})_i$, which is a contradiction.

$\square$

10.2 A generalization of the Rawlsian assignment

Let σ be an ordering of the set of integers $\{2, \dots, n\}$, and let Σ denote the set of all such orderings. We denote the i-th element of σ by σ_i.[5] Similar to Section 4, given an assignment x, an ordering $\sigma \in \Sigma$, and the vectors $(b_i^x)_{i\in I}$ containing the cumulative assignment probabilities, we define the vector $B^{x,\sigma} \in [0,1]^{(n-1)^2}$ as follows.

1. The first elements $(B_1^{x,\sigma}, \dots, B_n^{x,\sigma})$ are the elements $(b_1^x(\sigma_1), \dots, b_n^x(\sigma_1)))$ listed in a non-increasing order.

2. Elements $(B_{n+1}^{x,\sigma}, \dots, B_{2n}^{x,\sigma})$ are the elements $(b_1^x(\sigma_2), \dots, b_n^x(\sigma_2)))$ listed in a non-increasing order.

3. In general, elements $(B_{(k-1)n+1}^{x,\sigma}, \dots, B_{kn}^{x,\sigma})$ for $k = 1, \dots, n-1$, are the elements $(b_1^x(\sigma_k), \dots, b_n^x(\sigma_k))$ listed in a non-increasing order.

That is, the first elements of $B^{x,\sigma}$ are the probabilities with which each agent receives the objects with a rank between σ_1 and n, then the probabilities with which each agent receives the objects with a rank between σ_2 and n and so on, and so forth. The Rawlsian assignment corresponds to B^{x,σ_R}, with $\sigma_R = (n, n-1, \dots, 2)$. Like in Section 4, given two assignments x and y, we compare the vectors $B^{x,\sigma}$ and $B^{y,\sigma}$ lexicographically.

Definition 8. *Given two assignments x and y and an ordering $\sigma \in \Sigma$, x σ-**dominates** y if there is $j \in \{1, \dots, (n-1)^2\}$ such that $B_j^{x,\sigma} < B_j^{y,\sigma}$, and for all $i < j$, $B_i^{x,\sigma} = B_i^{y,\sigma}$.*

[5] Note that we do not include the first preference in the orderings in Σ, because the total assignment probability of each agent always equals one.

Definition 9. *Given an ordering $\sigma \in \Sigma$, an assignment x is* **$\boldsymbol{\sigma}$-minimal** *if it is not σ-dominated by any other assignment.*

Similar to the results in Section 5, a σ-minimal assignment is unique and sd-efficient, for any ordering $\sigma \in \Sigma$.

Proposition 7. *Each problem has a unique σ-minimal assignment, for any order $\sigma \in \Sigma$.*

Proof. Given an ordering $\sigma \in \Sigma$, suppose x and y are two different assignments that are both σ-minimal, and have the same vector B^σ. Following a similar reasoning as the proof of Proposition 1, we can show that the assignment $\frac{1}{2}(x+y)$ σ-dominates both x and y, contradicting the fact that x and y are both σ-minimal assignments. $\square$

Proposition 8. *For any ordering $\sigma \in \Sigma$, the σ-minimal assignment is sd-efficient.*

Proof. Given any ordering $\sigma \in \Sigma$, suppose that assignment x is σ-minimal, but not sd-efficient. Then, there is an improving cycle $(1, o_1, 2, o_2, \ldots, K, o_K)$. We can assume wlog that $o_i \neq o_j$ for every i, j. Denote by $\epsilon > 0$ the minimum of all the probabilities: $\epsilon = \min_{k=1,\ldots,K} x_{ko_k}$. Implement the cycle by decreasing each probability x_{ko_k}, for $k = 1, \ldots, K$, by ϵ, and increasing x_{1o_K}, and $x_{ko_{k-1}}$, $k = 2, \ldots, K$ by the same share. We get a new random assignment y. We will show that it σ-dominates x, which is a contradiction.

Because each agent is better off after we implement the improvement cycle, we know that $r_{ko_{k-1}} < r_{ko_k}$, for each $k = 1, \ldots, K$ (we use the notation $o_0 = o_K$). This implies that, for each agent $k = 1, \ldots, K$, the cumulative probability of being assigned to an object ranked $r_{ko_{k-1}}$-th or worse is equal in x and y, i.e.,

$$b_k^y(r_{ko_{k-1}}) = b_k^x(r_{ko_{k-1}}) + \epsilon - \epsilon = b_k^x(r_{ko_{k-1}}).$$

Similarly, the cumulative probability of being assigned to an object ranked r_{ko_k}-th or worse is lower in y than in x, i.e.,

$$b_k^y(r_{ko_k}) = b_k^x(r_{ko_k}) - \epsilon \iff b_k^y(r_{ko_k}) < b_k^x(r_{ko_k}).$$

As a result, each element of the vector $B^{y,\sigma}$ associated with assignment y is not larger than the corresponding element in the vector $B^{x,\sigma}$ associated to assignment x. Moreover, there are at least K elements of $B^{y,\sigma}$ that are strictly smaller than the corresponding elements in $B^{x,\sigma}$. Hence, y σ-dominates x. $\square$

We define the σ-minimal rule as the rule that assigns the σ-minimal assignment to each problem. As for the Rawlsian rule, the σ-minimal rule satisfies anonymity.

Proposition 9. *The σ-minimal rule satisfies anonymity, for any ordering $\sigma \in \Sigma$.*

Proof. The definition of vector $B^{x,\sigma}$ does not depend on agents' identities, only on the probabilities of the assignment x and the order σ. Thus, the same proof as in Proposition 2 applies in this generalized setting. $\square$

10.3 Relation with fractional Boston assignment

The fractional Boston rule by Bogomolnaia (2015), which was further studied by Chen, Harless, and Jiao (2023), is the outcome of the following procedure.

- In Step 1, all agents start by "consuming" their most preferred object simultaneously at equal speeds. An agent stops consuming when the object is exhausted. Step 1 ends by removing all agents who are assigned to their first-ranked object with probability one.
- In Step k, all remaining agents consume from their k-th preferred object. An agent stops consuming when the object is exhausted, or when the sum of the assignment probabilities of the agent equals one. Step k ends by removing the agents for whom the sum of the assignment probabilities equals one.

The algorithm ends when the last agent is removed.

While both the fractional Boston rule and the σ^B-Rawlsian assignment for $\sigma^B = (2, 3, \ldots, n)$ start by lexicographically maximizing the probabilities of being assigned to the most preferred object, the following example shows that they are not identical.

Remark 1. *The outcome of the fractional Boston rule does not correspond to the σ_B-minimal assignment for $\sigma^B = (2, 3, \ldots, n)$.*

Proof. Consider the following problem with 7 agents:

$$\begin{array}{c|cccc}
\succeq_1 & a & b & e & \ldots \\
\succeq_2 & a & b & c & \ldots \\
\succeq_3, \succeq_4 & c & b & a & \ldots \\
\succeq_5, \succeq_6, \succeq_7 & d & a & e & \ldots
\end{array}$$

The assignment probabilities for the three most preferred objects of each agent in the σ^B-minimal assignment and in the fractional Boston assignment are (element x_{ij} is the probability with which agent i is assigned to her j most preferred object):

$$x^{\sigma^B} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \mathbf{0} & \dots \\ \frac{1}{2} & \frac{1}{4} & 0 & \dots \\ \frac{1}{2} & \frac{1}{4} & 0 & \dots \\ \frac{1}{2} & \frac{1}{4} & 0 & \dots \\ \frac{1}{3} & 0 & \mathbf{\frac{1}{3}} & \dots \\ \frac{1}{3} & 0 & \mathbf{\frac{1}{3}} & \dots \\ \frac{1}{3} & 0 & \mathbf{\frac{1}{3}} & \dots \end{pmatrix}, \quad x^{FB} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \mathbf{\frac{1}{4}} & \dots \\ \frac{1}{2} & \frac{1}{4} & 0 & \dots \\ \frac{1}{2} & \frac{1}{4} & 0 & \dots \\ \frac{1}{2} & \frac{1}{4} & 0 & \dots \\ \frac{1}{3} & 0 & \mathbf{\frac{1}{4}} & \dots \\ \frac{1}{3} & 0 & \mathbf{\frac{1}{4}} & \dots \\ \frac{1}{3} & 0 & \mathbf{\frac{1}{4}} & \dots \end{pmatrix}.$$

The difference in the assignment probabilities for the object of third choice comes from the following reasoning. It can be shown that the σ^B-minimal assignment can be obtained by first lexicographically maximizing the vector of the probabilities with which the agents are assigned to the object of their first choice, then to lexicographically maximize the vector with the probabilities with which the agents are assigned to the objects of their first or second choice, etc. As such, after having fixed the probabilities of being assigned to each agent's two most preferred objects, the σ^B-minimal assignment maximizes the vector of the cumulative probabilities of being assigned to one of the three most preferred objects, and, therefore, divides object e equally among agents 5,6, and 7. In the fractional Boston assignment, however, at the beginning of the thirds step, agents 1,5,6, and 7 start eating with equal speeds from object e, which allows agent 1 to consume one quarter of the object. Thus, in the σ^B-minimal assignment the minimum probability of being assigned to the three most preferred objects is $\frac{2}{3}$, while it is $\frac{7}{12} < \frac{2}{3}$ in the fractional Boston assignment.

□

10.4 Relation with rank efficiency

Given an assignment x, we define $M^x(k) = \sum_{i \in I} b_i^x(k)$ for $k = 1, \dots, n$. So, for example, $M^x(n)$ is the sum of the probabilities with which each agent is assigned her least preferred option. Equivalently, $M^x(k)$ is the expected number of agents who receive an object ranked in position k or lower at x.

Definition 10. *An assignment, x, is said to* **rank-dominate** *another assignment, y, if the rank distribution of y first-order stochastically dominates that of x, that is, if:*

$$M_y(k) \geq M_x(k).$$

for all ranks, k, with a strict inequality for at least one k.

If an assignment is not rank-dominated by any other assignment, it is **rank efficient***.*

The previous definition is equivalent to the original of Featherstone (2020), where the sum of the probabilities are computed from the most preferred object to k. The following example shows a problem where the Rawlsian assignment is not rank-efficient.

Remark 2. *A rank efficient assignment may not be Rawlsian.*

Proof. Consider the following problem:

$$\begin{array}{c|ccccc} \succeq_1 & a & \boxed{d} & c & b & e \\ \succeq_2 & \boxed{b} & c & a & d & e \\ \succeq_3 & \boxed{c} & b & a & d & e \\ \succeq_4 & b & \boxed{a} & d & c & e \\ \succeq_5 & b & a & \boxed{d} & e & c \end{array}$$

The following assignment is rank efficient:

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

but it is not Rawlsian. Indeed, its associated vector is

$$B^y = (0,0,0,0,0;1,0,0,0,0;1,1,0,0,0;1,1,0,0,0;1,1,1,1,1).$$

Consider the following assignment x (boxed in agents' preferences):

$$x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that x R-dominates y. $\square$

Remark 3. *A Rawlsian assignment may not be rank-efficient.*

Proof. Consider the following problem:

$$\begin{array}{c|ccc} \succeq_1 & a & b & c \\ \succeq_2 & a & b & c \\ \succeq_3 & b & a & c \end{array}$$

The following is the Rawlsian assignment of the problem:

$$x = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Consider the following assignment:

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Compute $M_x(k)$ and $M_y(k)$:

	$M_j(3)$	$M_j(2)$	$M_j(1)$
x	1	$\frac{4}{3}$	3
y	1	1	3

Assignment y Rank-dominates assignment x.

□

10.5 Relation with egalitarian assignments

In this section, we define the concept of egalitarian assignments (Duddy, 2022), and show that it is independent from the concept of Rawlsian assignments. Let $t^k(\succ_i, x_i)$ be the probability with which agent i receives her top k objects under allocation x_i.

Definition 11. *An assignment x is* **egalitarian** *if there does not exist another assignment y and an agent j such that:*

$$t^k(\succ_i, y_i) \geq t^k(\succ_j, x_j),$$

for all i and $k = 1, \ldots, n$ (with the inequality being strict at least for one k, for every i).

Remark 4. *An egalitarian assignment may not be Rawlsian.*

Proof. Consider the following problem.

$$\begin{array}{c|cccc} \succeq_1 & a & b & c & d \\ \succeq_2 & b & c & a & d \\ \succeq_3 & a & b & c & d \\ \succeq_4 & b & a & d & c \end{array}$$

The following assignment is egalitarian:

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If not, there is another assignment y' such that agent 4 is assigned to object d with probability less than 1 (all agents in y are assigned to an object in $3rd$ position or higher). But in that case, one of the other agents, let's say $i \neq 4$, is assigned to object d with positive probability. Then,

$$t^3(\succ_i, y'_i) < 1 = t^3(\succ_I, y_I), \forall I,$$

The Rawlsian assignment of this problem is:

$$x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

Remark 5. *A Rawlsian assignment may not be egalitarian. Indeed, Duddy (2022) has an example in Section 4.5 showing that the Rawlsian assignment does not satisfy the egalitarian criterion.*

10.6 Relation with obvious manipulability

In this section, we extend the definition of obvious manipulability by Troyan and Morrill (2020) from a setting of deterministic assignments to a setting of probabilistic assignments.

First, we introduce the definition in a deterministic setting, as stated in Troyan (2022). Denote by M the set of feasible deterministic assignments (the set of $n \times n$ bi-stochastic matrices of which all elements are either zero or one). Let m_i denote the object to which agent $i \in I$ is assigned by assignment $m \in M$. A deterministic rule μ is a function that maps every problem to **a set of** deterministic assignments: for every $\succeq$, $\mu(\succeq) \subseteq M$.

Given a set of deterministic assignments $S \subseteq M$, let $\overline{\rho}_i(S) = \max_{m \in S} r_i(m_i)$ denote the rank of the least preferred object assigned to agent i by the assignments in S. Equivalently, let $\underline{\rho}_i(S) = \min_{m \in S} r_i(m_i)$ denote the rank of the most preferred object assigned to agent i by the assignments in S.

Definition 12. *A deterministic rule μ is not obviously manipulable if, for any agent i with preference profile $\succeq_i$ and any manipulation $\succeq'_i \neq \succeq_i$, the following two conditions hold:*

(i) $\max_{\succeq_{-i}} \overline{\rho}_i(\mu(\succeq_i, \succeq_{-i})) \leq \max_{\succeq_{-i}} \overline{\rho}_i(\mu(\succeq_i', \succeq_{-i}))$

(ii) $\min_{\succeq_{-i}} \underline{\rho}_i(\mu(\succeq_i, \succeq_{-i})) \leq \min_{\succeq_{-i}} \underline{\rho}_i(\mu(\succeq_i', \succeq_{-i}))$.

If either (i) or (ii) is violated for some agent i with preference profile $\succeq_i$, then we call $\succeq_i'$ an obvious manipulation for i, and say that the deterministic rule μ is obviously manipulable.

To extend this definition to probabilistic rules, as defined in Section 3, it suffices to redefine the notions of the worst-case rank $\overline{\rho}_i(S)$ and the best-case rank $\underline{\rho}_i(S)$ for probabilistic assignments. Given a probabilistic assignment $x \in X$, let $\overline{\omega}_i(x) = \max_{o \in O: x_{io} > 0} r_i(o)$ denote the rank of the least preferred object to which agent i is assigned by assignment x **with a strictly positive probability**. Equivalently, let $\underline{\omega}_i(x) = \min_{o \in O: x_{io} > 0} r_i(o)$ denote the rank of the most preferred object to which agent i is assigned by assignment x with a strictly positive probability. Note that for any representation of x as a lottery over the set of deterministic matchings, it holds that $\overline{\omega}_i(x) = \overline{\rho}_i(S)$ and $\underline{\omega}_i(x) = \underline{\rho}_i(S)$ for any agent i, where $S \subseteq M$ denotes the subset of the deterministic assignments that are given a strictly positive probability in the representation of x.

We can now define obvious manipulability for rules that output a probabilistic assignment as follows:

Definition 13. *A rule ϕ is not* **obviously manipulable** *if, for any agent i with preference profile $\succeq_i$ and any manipulation $\succeq_i' \neq \succeq_i$, the following two conditions hold:*

(i) $\max_{\succeq_{-i}} \overline{\omega}_i(\phi(\succeq_i, \succeq_{-i})) \leq \max_{\succeq_{-i}} \overline{\omega}_i(\phi(\succeq_i', \succeq_{-i}))$

(ii) $\min_{\succeq_{-i}} \underline{\omega}_i(\phi(\succeq_i, \succeq_{-i})) \leq \min_{\succeq_{-i}} \underline{\omega}_i(\phi(\succeq_i', \succeq_{-i}))$.

If either (i) or (ii) is violated for some agent i with preference profile $\succeq_i$, then we call $\succeq_i'$ an obvious manipulation for i, and rule ϕ is obviously manipulable.

In Example 5, we observe that agent 2 can decrease the worst-case rank of the objects to which he is assigned with a strictly positive probability by manipulating, as $\overline{\omega}_2(x) = 2$ while $\overline{\omega}_2(y) = 1$. Thus, condition (i) of Definition 13 is violated and the Rawlsian rule is obviously manipulable.

10.7 Relation with strategyproofness axioms by Mennle and Seuken (2021)

Mennle and Seuken (2021) show that a rule is strategyproof if and only if it satisfies the axioms of swap monotonicity, upper invariance and lower invariance. Following the result by Carroll (2012), the following three axioms only consider misreports in which the order of two consecutively ranked objects in a preference list is swapped. We use the notation $a > b$ to denote that objects a and b are ranked consecutively in a preference list, and that a is strictly preferred to b.

Definition 14. *A rule ϕ is* **swap monotonic** *if, for all agents $i \in I$, all preference profiles $(\succeq_i, \succeq_{-i})$, and all misreports $\succeq_i'$ where $a > b$ at $\succeq_i$, and $b > a$ at $\succeq_i'$, one of the following two conditions holds:*

- *either: $\phi_i(\succeq_i, \succeq_{-i}) = \phi_i(\succeq_i', \succeq_{-i})$,*
- *or: $\phi_{i,b}(\succeq_i', \succeq_{-i}) > \phi_{i,b}(\succeq_i, \succeq_{-i})$.*

Definition 15. *A rule ϕ is* **upper invariant** *if, for every agent $i \in I$, all preference profiles $(\succeq_i, \succeq_{-i})$, and all misreports $\succeq_i'$ where $a > b$ at $\succeq_i$, and $b > a$ at $\succeq_i'$, it holds that $\phi_{ij}(\succeq_i, \succeq_{-i}) = \phi_{ij}(\succeq_i', \succeq_{-i})$ for all objects j for which $j \succeq_i a$.*

Definition 16. *A rule ϕ is* **lower invariant** *if, for all agents $i \in I$, all preference profiles $(\succeq_i, \succeq_{-i})$, and all misreports $\succeq_i'$ where $a > b$ at $\succeq_i$, and $b > a$ at $\succeq_i'$, it holds that $\phi_{ij}(\succeq_i, \succeq_{-i}) = \phi_{ij}(\succeq_i', \succeq_{-i})$ for all objects j for which $b \succeq_i j$.*

The intuition behind Definition 14 is that a rule is swap monotonic if increasing the preference for an object b either increases the probability of being assigned to object b, or does not affect the agent's assignment probabilities at all. Moreover, a rule is upper (resp. lower) invariant if swapping the order of two consecutively ranked objects a and b does not affect the assignment probabilities of the objects that are more preferred than a (resp. less preferred than b).

Remark 6. *The Rawlsian rule satisfies lower invariance.*

Proof. Let x denote the Rawlsian assignment for true preferences $\succeq$, and let y denote the Rawlsian assignment for preferences $(\succeq_i', \succeq_{-i})$, such that $a > b$ at $\succeq_i$, and $b > a$ at $\succeq_i'$. Suppose that the Rawlsian rule does not satisfy lower invariance. Then, the assignment probabilities of agent i for the objects that she ranks lower than b in $\succeq_i$ differ in x and y. Let $B^x(\succeq)$ denote vector associated with assignment x for preferences $\succeq$.

Because y is the Rawlsian assignment for preferences $(\succeq_i', \succeq_{-i})$, y R-dominates x for these preferences. Since the assignment probabilities for the objects that agent i ranks lower than b in $\succeq_i$ differ in x and y, this means that vector $B^y(\succeq_i', \succeq_{-i})$ lexicographically dominates vector $B^x(\succeq_i', \succeq_{-i})$ in one of the first $n(n - r_{ib})$ elements, where r_{ib} is the rank of b in $\succeq_i$.

However, as $\succeq_i$ and $\succeq_i'$ are identical for the objects that are ranked lower than b in $\succeq_i$, it holds that $B_j^y(\succeq_i', \succeq_{-i}) = B_j^y(\succeq)$ for the first $j \leq n(n - r_{ib})$ elements, and the same is true for $B^x(\succeq_i', \succeq_{-i})$ and $B^x(\succeq)$. Therefore, $B^y(\succeq)$ should also lexicographically dominate $B^x(\succeq)$ in one of the first $n(n - r_{ib})$ elements, which is in contradiction with the fact that x is the Rawlsian assignment for preferences $\succeq$. Hence, the Rawlsian rule satisfies lower invariance. $\square$

Remark 7. *The Rawlsian rule violates swap monotonicity and upper invariance.*

Proof. Consider the following preferences.

$$\begin{array}{c|ccc} \succeq_1 & c & a & b \\ \succeq_2 & b & c & a \\ \succeq_3 & b & c & a \end{array}$$

The Rawlsian assignment for these preferences is

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

If agent 1 alternatively reveals $\succeq_1' = (c, b, a)$, swapping the order of objects a and b, the Rawlsian assignment becomes

$$x' = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}.$$

First, the probability of being assigned to object b for agent 1 remains the same, while the probability of being assigned to object a decreases, which violates the swap monotonicity axiom.

Second, swapping the order of objects a and b causes the probability of agent 1 being assigned to object c to decrease, which violates the upper invariance axiom. □

10.8 Rawlsian assignments and cardinal utilities

In this section we first prove in Proposition 10 that if x is an sd-efficient assignment, there exists a cardinal representation of the agents' ordinal preferences such that x maximizes the expected utility of the worst-off agent, among all possible assignments. Next, we prove in Proposition 4 that the reverse is not true, and that an instance $\succeq$ may admit a continuum of different assignments that maximize the utility of the worst-off agent for some cardinal utilities, even when all agents have the same cardinal utilities.

Proposition 10. *Consider a problem $\succeq$ and an sd-efficient assignment x for problem $\succeq$. Then, there exists a cardinal representation of $\succeq$, denoted by $U = (U_i)_{i \in I}$, such that:*

$$x \in \arg\max_{x' \in X} \{\min_{i \in I} U_i(x')\}.$$

Proof. Consider an sd-efficient assignment x. By definition, x is not stochastically dominated by any other assignment. Equivalently, for **any** cardinal representation of the agents' preferences, there is no assignment such that the expected utility of every agent is greater than or equal to the expected utility in this assignment (strictly greater for one agent). Fix another cardinal representation of the agents' preferences such that the expected utility of each agent under x is the same. Then, the minimum cardinal utility in x is the utility of any agent. Suppose there is another assignment y such that the minimum expected utility among all agents is strictly greater than the minimum expected utility under x. Therefore, every agent's expected utility under y is greater than her expected utility under x (because it is greater than or equal to the minimum expected utility). But this implies that x is not sd-efficient, which is a contradiction. $\square$

Proof of Proposition 4.

Proof. Consider the following example:

$\succsim_1$	$\succsim_2$	$\succsim_3$
a	a	b
b	b	a
c	c	c

And consider the following assignment with $\alpha \in [0, \frac{1}{2}]$:

$$x^\alpha = \begin{pmatrix} 0.5 & 0.5-\alpha & \alpha \\ 0.5 & 0.5-\alpha & \alpha \\ 0 & 2\alpha & 1-2\alpha \end{pmatrix}.$$

Note that x^α is indeed a feasible assignment that treats equals equally. Moreover, $x^\alpha_{1a} = x^\alpha_{2a} = 0.5$ and $x^\alpha_{3a} = 0$ holds because otherwise agent 3 could decrease the probability of being assigned object a while agents 1 and 2 could increase their probability of being assigned object a. This exchange would increase everyone's utilities, and any alternative assignment than x^α is therefore not an assignment that maximizes the utility of the worst-off agent.

Consider the case where agents' cardinal utilities for being assigned their first, second, and third object are the same, that is, $U_1(a) = U_2(a) = U_3(b)$, $U_1(b) = U_2(b) = U_3(a)$, and $U_1(c) = U_2(c) = U_3(c)$. Then, the utilities that the agents experience in x^α are:

$$U_1(\alpha) = U_2(\alpha) = 0.5 \cdot u(1) + (0.5-\alpha)\cdot u(2) + \alpha \cdot u(3)$$
$$U_3(\alpha) = 2\alpha \cdot u(1) + (1-2\alpha)\cdot u(3)$$

Because both functions are linear in α, we maximize the utility of the worst-off agent by determining α^* for which $U_1(\alpha^*) = U_2(\alpha^*) = U_3(\alpha^*)$. It can be checked that we satisfy this condition for

$$\alpha^* = \frac{0.5\cdot u(1) + 0.5\cdot u(2) - u(3)}{2\cdot u(1) + u(2) - 3\cdot u(3)}.$$

Because $u(1) \geq u(2) \geq u(3)$, it must hold that $\frac{1}{4} \leq \alpha^* \leq \frac{1}{3}$. Hence, for each assignment x^α with $\alpha \in [\frac{1}{4}, \frac{1}{3}]$, there exist cardinal utility functions for which x^α maximizes the utility of the worst-off agent. Note that $x^{\frac{1}{3}}$ is the Rawlsian assignment. $\square$

10.9 The MTAV rule.

In this section, we define the MTAV following Paleo (2021). Let $\mathcal{M}$ be the set of deterministic assignments. Each assignment in $\mathcal{M}$ is represented by a $n \times n$ matrix M where $m_{ij} = 1$ if, and only if, agent i receives object j. Given two matrices M and M' in $\mathcal{M}$, we denote by $M \odot M'$ the matrix where each element is the product of the corresponding elements of M and M': $(M \odot M')_{ij} = (m_{ij})(m'_{ij})$. Also, denote by $\max(M)$ and $sum(M)$ the maximium and the sum of the elements of M, respectively.

Given a problem $(\succeq)$, define the matrix P with agents' preferences, where P_{ij} is the rank of object j in i's preferences (equivalently, $P_{ij} = r_{ij}$). The **MTAV** is defined as follows.

1. For each $M \in \mathcal{M}$ computes $P \odot M$.
2. Compute $\max(P \odot M)$.
3. Select the assignments that minimize $\max(P \odot M)$.
4. Among the assignments selected in the previous step, select those that minimize $sum(P \odot M)$.
5. If multiple assignments are selected in the last step, take one assignment at random.

To implement MTAV, we have used its publicly available code (`https://github.com/eze91/MTAV`). In the last step, this code will simply generate one of the multiple assignments, rather than generating all of them and selecting each with equal probability. Note that both approaches are not equivalent, as discussed in detail in Demeulemeester, Goossens, Hermans, and Leus (2023).

10.10 Proof of Proposition 5

Proof. Suppose there exists a constant L such that $\lim_{n\to+\infty} \mathbb{E}(r_{max}^{Rawls}) = L$. This is equivalent to saying that, in expectation, all agents only receive strictly positive probabilities for their L most-preferred objects when $n \to \infty$.

Consider the cost matrix C such that being assigned to the top object has a cost of 1, being assigned to the next object has a cost of 2, etc. When an assignment only considers the L most-preferred objects of each agents, the cost of this assignment is at least $(n-1)\cdot 1 + L$. This cost is obtained when all but one of the agents are assigned, in expectation, to their first choice, and one agent is assigned, in expectation to her L-th choice.[6] By definition of the minimum,

[6] Note that when the Rawlsian assignment has a maximum rank of L for a given preference profile, then the sum of the assignment probabilities of the agents to their L-th most-preferred object will always be an integer at least equal to 1. Consider the object with the lowest rank in any agent's preference list for which they receive a strictly positive probability by the Rawlsian assignment. Assume, for contradiction, that this object is not fully divided among agents who rank this object L-th, and that there exists an agent who ranks that object strictly better than L-th who receives a strictly positive probability for that object. In that case, we can fully assign the object to that agent, which would result in an assignment that Rawlsian-dominates the original assignment, which leads to a contradiction.

we know that

$$\min_{\pi \in X} \frac{1}{n} \sum_{i \in I} C_{i\pi(i)} \leq \frac{n-1+L}{n}.$$

Taking the limit of this expression, we find that

$$\lim_{n\to\infty} \min_{\pi \in X} \frac{1}{n} \sum_{i \in I} C_{i\pi(i)} \leq \lim_{n\to\infty} \frac{n-1+L}{n} = 1.$$

However, Parviainen (2004) showed that $\lim_{n\to\infty} \min_{\pi\in X} \frac{1}{n} \sum_{i\in I} C_{i\pi(i)} \geq \frac{\pi^2}{6}$, which leads to a contradiction. □

10.11 Proof of Proposition 6.

Proof. Consider a complete bipartite graph (I, O, E) where one set of nodes I is the set of agents, and the other set of nodes O is the set of objects. There is an edge in E between an agent and an object if the agent ranks the object. When we choose exactly k edges for each agent, this is the same as considering the k most preferred objects of each agent.

For a given preferences $\succeq$, denote by $\tilde{\mathcal{G}}(n, k)$ the resulting bipartite graph when we consider the k most preferred objects of each agent. The maximum rank of the Rawlsian assignment, r_{max}^{Rawls}, is equal to some value $k \in \mathbb{N}$ when there exists a perfect matching in $\tilde{\mathcal{G}}(n, k)$, but not in $\tilde{\mathcal{G}}(n, k-1)$.

Additionally, let $\tilde{X}_{n,k}$ be the event that $\tilde{\mathcal{G}}(n, k)$ contains a perfect matching, i.e.,

$$\tilde{X}_{n,k} = \begin{cases} 1 & \text{if } \tilde{\mathcal{G}}(n,k) \text{ has a perfect matching,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, the probability that k is the maximum rank of the Rawlsian assignment equals:

$$\mathbb{P}(r_{max}^{\text{Rawls}} = k) = [1 - \mathbb{P}(\tilde{X}_{n,k-1} = 1)] \cdot \mathbb{P}(\tilde{X}_{n,k} = 1).$$

Note that the $\mathbb{P}(\tilde{X}_{n,k-1} = 1) \cdot \mathbb{P}(\tilde{X}_{n,k} = 1)$ is the probability that there is a perfect matching when we consider the first $k-1$ objects of each agent and there is a perfect matching when we consider the first k objects. But, if there is perfect matching for $k-1$, there is also for k. So:

$$\mathbb{P}(\tilde{X}_{n,k-1} = 1) \cdot \mathbb{P}(\tilde{X}_{n,k} = 1) = \mathbb{P}(\tilde{X}_{n,k-1} = 1).$$

Thus:

$$\mathbb{P}(r_{max}^{\text{Rawls}}) = \mathbb{P}(\tilde{X}_{n,k} = 1) - \mathbb{P}(\tilde{X}_{n,k-1} = 1). \tag{3}$$

Additionally, denote by $\mathcal{G}(n, p)$ the random bipartite graph with n nodes in each set, and in which each edge is independently selected with probability p. Similarly to $\tilde{\mathcal{G}}(n, k)$, let $X_{n,p}$

denote the event that $\mathcal{G}(n,p)$ contains a perfect matching.

Denote the minimum degree of a graph $\mathcal{G}$ by $\delta(\mathcal{G})$. Erdős and Rényi (1966) and Erdős and Rényi (1968) showed for various types of graphs that, in the limit (when $n \to +\infty$), the probability that a perfect matching exists is equal to the probability that the minimum degree of the graph is at least one (no isolated vertices). Because half of the nodes in $\tilde{\mathcal{G}}(n,k)$ have a guaranteed degree of k, the probability of having a minimum degree of at least one is not smaller in $\tilde{\mathcal{G}}(n,k)$ than in $\mathcal{G}(n,\frac{k}{n})$. As a consequence, it holds that, for every $k \in \{1,\dots,n\}$:

$$\lim_{n\to\infty} \mathbb{P}(\tilde{X}_{n,k}=1) = \lim_{n\to\infty} \mathbb{P}(\delta(\tilde{\mathcal{G}}(n,k)) \geq 1) \geq \lim_{n\to\infty} \mathbb{P}(\delta(\mathcal{G}(n,\frac{k}{n}) \geq 1)) = \lim_{n\to\infty} \mathbb{P}(X_{n,\frac{k}{n}}=1). \tag{4}$$

The expected maximum rank of the Rawlsian assignment is equal to:

$$\mathbb{E}(r_n^{max}) = \sum_{k=1}^{n} k \cdot \mathbb{P}(r_n^{max}=k). \tag{5}$$

By applying Equation (3), we obtain that

$$\mathbb{E}(r_{max}^{\text{Rawls}}) = \sum_{k=1}^{n} k \cdot \left(\mathbb{P}(\tilde{X}_{n,k}=1) - \mathbb{P}(\tilde{X}_{n,k-1}=1)\right) \tag{6}$$

$$= n - \sum_{k=1}^{n-1} \mathbb{P}(\tilde{X}_{n,k}=1), \tag{7}$$

where $\mathbb{P}(\tilde{X}_{n,0}=1)=0$, and the second equality follows from the fact that $\mathbb{P}(\tilde{X}_{n,n}=1)=1$.

We are interested in evaluation this expression in the limit. By applying Equation (4), we can bound the limit of the expected maximum rank by the Rawlsian assignment in terms of the probability that a perfect matching exists in the well-studied class of bipartite random graphs $\mathcal{G}(n,p)$ in which each edge is selected with uniform probability p.

$$\lim_{n\to\infty} \mathbb{E}(r_{max}^{\text{Rawls}}) = \lim_{n\to\infty} \left(n - \sum_{k=1}^{n-1} \mathbb{P}(\tilde{X}_{n,k}=1)\right) \tag{8}$$

$$\leq \lim_{n\to\infty} \left(n - \sum_{k=1}^{n-1} \mathbb{P}(X_{n,\frac{k}{n}}=1)\right). \tag{9}$$

After some manipulations, the expression of which we take the limit can be written as:

$$n - \sum_{k=1}^{n-1} \mathbb{P}(X_{n,\frac{k}{n}} = 1) = n - \sum_{k=1}^{n-1} (1 - (1 - \mathbb{P}(X_{n,\frac{k}{n}} = 1))) \tag{10}$$

$$= n - \sum_{k=1}^{n-1} 1 + \sum_{k=1}^{n-1} (1 - \mathbb{P}(X_{n,\frac{k}{n}} = 1)) \tag{11}$$

$$= 1 + \sum_{k=1}^{n-1} (1 - \mathbb{P}(X_{n,\frac{k}{n}} = 1)) \tag{12}$$

We know from Erdős and Rényi (1968) that

$$\lim_{n\to\infty} (1 - \mathbb{P}(X_{n,\frac{\ln(n)+c_n}{n}} = 1)) = \begin{cases} 1 & \text{if } c_n \to -\infty \\ 1 - e^{-2e^{-c}} & \text{if } c_n \to c \\ 0 & \text{if } c_n \to \infty \end{cases} . \tag{13}$$

We can then rewrite Equation (9) as

$$\lim_{n\to\infty} \mathbb{E}(r_{max}^{\text{Rawls}}) \leq 1 + \lim_{n\to\infty} \left(\sum_{k=1-\ln(n)}^{n-1-\ln(n)} (1 - \mathbb{P}(X_{n,\frac{\ln(n)+k}{n}} = 1)) \right) \tag{14}$$

$$= 1 + \lim_{n\to\infty} \left(\sum_{k=1-\ln(n)}^{\lfloor \ln(n) \rfloor - \ln(n) - 1} (1 - \mathbb{P}(X_{n,\frac{\ln(n)+k}{n}} = 1)) + \sum_{k=\lfloor \ln(n) \rfloor - \ln(n)}^{n-1-\ln(n)} (1 - \mathbb{P}(X_{n,\frac{\ln(n)+k}{n}} = 1)) \right) \tag{15}$$

Note that these expressions are a bit unfamiliar because the indices of the summations are not integer: the interpretation is that, starting from the lower index, we sum for values of k that increase with step size one.

If we can show that the limit of the first summation is well-defined, and that second summation is a constant, we can rewrite this expression as:[7]

$$1 + \lim_{n\to\infty} \left(\sum_{k=1-\ln(n)}^{\lfloor \ln(n) \rfloor - \ln(n) - 1} (1 - \mathbb{P}(X_{n,\frac{\ln(n)+k}{n}} = 1)) \right) + \lim_{n\to\infty} \left(\sum_{k=\lfloor \ln(n) \rfloor - \ln(n)}^{n-1-\ln(n)} (1 - \mathbb{P}(X_{n,\frac{\ln(n)+k}{n}} = 1)) \right) . \tag{16}$$

Because probabilities are non-negative, an easy bound on the limit of the first summation

[7] In fact, we will show that this expression is similar to $\lim_{n\to\infty}(\ln(n) + constant) = \lim_{n\to\infty}(\ln(n)) + constant$.

would simply be:

$$\sum_{k=1-\ln(n)}^{\lfloor \ln(n) \rfloor - \ln(n) - 1} (1 - \mathbb{P}(X_{n, \frac{\ln(n)+k}{n}} = 1)) \leq \sum_{k=1-\ln(n)}^{\lfloor \ln(n) \rfloor - \ln(n) - 1} 1$$
$$= \lfloor \ln(n) \rfloor - \ln(n) - 1 - (1 - \ln(n)) + 1 = \lfloor \ln(n) \rfloor - 1. \tag{17}$$

For the limit of the second summation, we can write that

$$\lim_{n\to\infty} \left(\sum_{k=\lfloor \ln(n) \rfloor - \ln(n)}^{n-1-\ln(n)} (1 - \mathbb{P}(X_{n, \frac{\ln(n)+k}{n}} = 1)) \right) \leq \lim_{n\to\infty} \left(\sum_{k=-1}^{n-1-\lceil \ln(n) \rceil} (1 - \mathbb{P}(X_{n, \frac{\ln(n)+k}{n}} = 1)) \right) \tag{18}$$

$$\leq \lim_{n\to\infty} \left(\sum_{k=-1}^{+\infty} (1 - \mathbb{P}(X_{n, \frac{\ln(n)+k}{n}} = 1)) \right) \tag{19}$$

Because $1 - \mathbb{P}(X_{n, \frac{\ln(n)+k}{n}} = 1)$ is a decreasing function in k, we find an upper bound on this expression by shifting the values for which this function is evaluated downwards with a positive value of $\lceil \ln(n) \rceil - \ln(n)$ (first inequality). Because $1 - \mathbb{P}(X_{n, \frac{\ln(n)+k}{n}} = 1)) \geq 0$, we obtain an upper bound by changing the ending index of the summation from $n - 1 - \lceil \ln(n) \rceil$ to $+\infty$ (second inequality).

Now observe from (13) that $\lim_{n\to\infty}(1 - \mathbb{P}(X_{n, \frac{\ln(n)+k}{n}} = 1)) = 1 - e^{-2e^{-k}}$ for a constant value of k. If we can show that

$$\sum_{k=-1}^{+\infty} 1 - e^{-2e^{-k}} \tag{20}$$

converges to a constant, then we can bring the limit inside the summation in Expression (19). We will use the ratio test to show the convergence of the series $a_k = 1 - e^{-2e^{-k}}$.

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{1 - e^{-2e^{-(k+1)}}}{1 - e^{-2e^{-k}}} \right| = \lim_{k\to\infty} \frac{1 - e^{-2e^{-(k+1)}}}{1 - e^{-2e^{-k}}}, \tag{21}$$

where the last equality holds because both the numerator and the denominator are positive. We can simplify the numerator as:

$$1 - e^{-2e^{-(k+1)}} = 1 - e^{\frac{-2e^{-k}}{e}} \tag{22}$$

For large values of k, both the numerator and the denominator go to zero. We can use

L'Hôpital's rule, which states that

$$\lim_{k\to\infty} \frac{1-e^{-2e^{-(k+1)}}}{1-e^{-2e^{-k}}} = \lim_{k\to\infty} \frac{\frac{d}{dk}\left(1-e^{-2e^{-(k+1)}}\right)}{\frac{d}{dk}\left(1-e^{-2e^{-k}}\right)} \tag{23}$$

$$= \lim_{k\to\infty} \frac{-2e^{-k-1}e^{-2e^{-k-1}}}{-2e^{-k}e^{-2e^{-k}}} \tag{24}$$

$$= \lim_{k\to\infty} \frac{e^{-k-1}}{e^{-k}} \cdot \frac{e^{-2e^{-k-1}}}{e^{-2e^{-k}}} \tag{25}$$

$$= \lim_{k\to\infty} e^{-1} \cdot e^{2(e^{-k}-e^{-k-1})}. \tag{26}$$

As e^{-1} is a constant term, and because the exponent $2(e^{-k}-e^{-k-1})$ goes to zero for large values of k, this is equivalent to:

$$\lim_{k\to\infty} e^{-1} \cdot e^{2(e^{-k}-e^{-k-1})} = e^{-1} \cdot \lim_{k\to\infty} e^{2(e^{-k}-e^{-k-1})} = e^{-1} \cdot e^{0} = e^{-1}. \tag{27}$$

Note that $\frac{1}{e} \approx 0.3679 < 1$. As a consequence, $\sum_{k=-1}^{+\infty} 1 - e^{-2e^{-k}}$ converges to a constant. As a result, we can rewrite Expression (15) as Expression (16).

Putting the bounds of Expressions (17) and (19) together in Expression (16), we obtain:

$$\lim_{n\to\infty} \mathbb{E}(r_{max}^{Rawls}) \leq 1 + \lim_{n\to\infty} \left(\sum_{k=1-\ln(n)}^{\lfloor \ln(n) \rfloor - \ln(n) - 1} (1 - \mathbb{P}(X_{n,\frac{\ln(n)+k}{n}} = 1)) \right)$$

$$+ \lim_{n\to\infty} \left(\sum_{k=\lfloor \ln(n) \rfloor - \ln(n)}^{n-1-\ln(n)} (1 - \mathbb{P}(X_{n,\frac{\ln(n)+k}{n}} = 1)) \right) \tag{28}$$

$$\leq \lim_{n\to\infty} \lfloor \ln(n) \rfloor + \sum_{k=-1}^{+\infty} \left(1 - e^{-2e^{-k}}\right) \tag{29}$$

$$\approx \lim_{n\to\infty} \lfloor \ln(n) \rfloor + 2.77026. \tag{30}$$

$\square$

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1 Descriptive Statistics

We first illustrate some basic characteristics of the cooperatives we analyze. Table 4 shows the number of families in each cooperative (which equals the number of apartments), and the (cumulative) number of different objects ranked in the top k positions ($k = 1, 2, 3, 4$) in absolute terms, and as percentage of the total number of objects.

	Size	1st	2nd	3rd	4th	1st	2nd	3rd	4th
C_1	26	16	20	21	22	61	76	80	84
C_2	18	9	11	13	14	50	61	72	77
C_3	4	3	4	4	4	75	100	100	100
C_4	4	2	2	4	4	50	50	100	100
C_5	28	17	23	24	25	60	82	85	89
C_6	8	5	8	8	8	62	100	100	100
C_7	29	16	22	24	25	55	75	82	86
C_8	12	6	9	11	11	50	75	91	91
C_9	15	9	11	13	15	60	73	86	100
C_{10}	4	3	3	3	4	75	75	75	100
C_{11}	11	6	7	10	10	54	63	90	90
C_{12}	16	11	14	15	15	68	87	93	93
C_{13}	39	13	23	28	31	33	58	71	79
C_{14}	42	19	24	27	28	45	57	64	66
C_{15}	14	5	8	9	10	35	57	64	71
C_{16}	6	3	4	5	5	50	66	83	83
C_{17}	9	6	8	9	9	66	88	100	100
C_{18}	15	10	10	12	12	66	66	80	80
C_{19}	9	4	5	5	5	44	55	55	55
C_{20}	20	8	11	15	17	40	55	75	85
C_{21}	24	16	21	23	23	66	87	95	95
C_{22}	7	4	7	7	7	57	100	100	100
C_{23}	40	17	26	30	34	42	65	75	85
C_{24}	8	3	5	5	5	37	62	62	62

Table 4: Size is the number of families in each cooperative. The next four columns, 1st, 2nd, 3rd, and 4th, have the cumulative number of different apartments ranked in the top 1, 2 ,3 and 4 positions, respectively. The last four columns express the previous four columns as a percentage of the total number of apartments.

2 Expected number of families that are assigned apartments with rank $k = 1, \ldots, n$ by the Rawlsian, PS and MTAV rules

Table 5: Expected number of families that are assigned apartments with rank up to $k = 1, \ldots, 16$ by the Rawlsian, PS and MTAV rules.

	C_1			C_2			C_3			C_4		
Position	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV
1	7.17	12.54	10	2.4	6.42	8	2	2.44	2	2	2	2
2	10.36	14.61	15	4.4	7.13	10	4	3.33	4	2	2	2
3	13.33	15.64	16	5	7.64	10	4	3.56	4	4	4	4
4	16.67	16.51	18	5	8.34	10	4	4	4	4	4	4
5	17	16.99	18	8	9.42	11	-	-	-	-	-	-
6	18	17.73	19	11	10.55	13	-	-	-	-	-	-
7	20	18.69	20	15	11.91	13	-	-	-	-	-	-
8	22	19.62	23	15	12.7	13	-	-	-	-	-	-
9	23	19.78	23	17	13.46	14	-	-	-	-	-	-
10	23	20.01	23	17	14.13	14	-	-	-	-	-	-
11	24	20.15	23	17	14.66	16	-	-	-	-	-	-
12	25	20.45	24	18	15.04	18	-	-	-	-	-	-
13	26	21.49	26	18	15.34	18	-	-	-	-	-	-
14	26	21.91	26	18	15.98	18	-	-	-	-	-	-
15	26	22.51	26	18	16.32	18	-	-	-	-	-	-
16	26	22.76	26	18	17.13	18	-	-	-	-	-	-
Total	26	26	26	18	18	18	-	-	-	-	-	-

Table 6: Expected number of families that are assigned apartments with rank up to $k = 1, \ldots, 16$ by the Rawlsian, PS and MTAV rules.

	C_5			C_6			C_7			C_8		
Position	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV
1	7.75	13.82	11	4	4.28	4	11.5	11.72	13	3.5	4.51	4
2	13.58	18.36	17	7	5.78	7	14.5	14.83	16	7.67	7.52	8
3	17	18.97	20	8	6.72	8	15.5	16.63	18	11	9.21	11
4	19	20.03	21	8	7.22	8	18	17.94	20	11	9.33	11
5	21.5	21.04	22	8	7.56	8	21.5	19.36	20	11	9.66	11
6	24.5	21.52	23	8	7.89	8	23	20.09	22	11	10.02	11
7	27	22.07	26	8	7.89	8	27	21.42	27	12	10.27	12
8	28	22.52	28	8	8	8	29	22.46	29	12	10.76	12
9	28	23.05	28	-	-	-	29	23.12	29	12	10.94	12
10	28	23.34	28	-	-	-	29	23.95	29	12	11.05	12
11	28	24.04	28	-	-	-	29	24.87	29	12	11.25	12
12	28	24.55	28	-	-	-	29	25.42	29	12	12	12
13	28	25.14	28	-	-	-	29	25.91	29	-	-	-
14	28	25.33	28	-	-	-	29	26.2	29	-	-	-
15	28	25.93	28	-	-	-	29	27.03	29	-	-	-
16	28	26.11	28	-	-	-	29	27.8	29	-	-	-
Total	28	28	28	-	-	-	29	29	29	-	-	-

Table 7: Expected number of families that are assigned apartments with rank up to $k = 1, \ldots, 16$ by the Rawlsian, PS and MTAV rules.

	C_9			C_{10}			C_{11}			C_{12}		
Position	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV
1	4.88	7.23	8	2.42	2.42	3	4	5.63	6	7.81	9.05	8
2	7.25	8.59	9	2.75	2.75	3	7	6.33	7	11.38	11.73	12
3	9	9.4	10	3	3	3	10	8.56	9	13	12.44	13
4	13	10.85	12	4	4	4	10	9.13	10	13.75	12.99	14
5	14	12.16	13	-	-	-	11	9.59	11	15	13.53	15
6	15	13.03	15	-	-	-	11	10.31	11	16	13.88	16
7	15	13.29	15	-	-	-	11	10.44	11	16	14.26	16
8	15	13.35	15	-	-	-	11	10.6	11	16	14.58	16
9	15	13.69	15	-	-	-	11	10.86	11	16	15.03	16
10	15	14.47	15	-	-	-	11	10.89	11	16	15.29	16
11	15	14.74	15	-	-	-	11	11	11	16	15.4	16
12	15	14.8	15	-	-	-	-	-	-	16	15.48	16
13	15	14.93	15	-	-	-	-	-	-	16	15.65	16
14	15	15	15	-	-	-	-	-	-	16	15.85	16
15	15	15	15	-	-	-	-	-	-	16	15.91	16
16	-	-	-	-	-	-	-	-	-	16	16	16
Total	-	-	-	-	-	-	-	-	-	-	-	-

Table 8: Expected number of families that are assigned apartments with rank up to $k = 1, \ldots, 16$ by the Rawlsian, PS and MTAV rules.

	C_{13}			C_{14}			C_{15}			C_{16}		
Position	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV
1	4.5	9.29	9	3.11	12	7	1.78	3.65	4	2.17	2.44	3
2	11.5	16.52	14	8.46	15.38	16	3.75	5.88	8	3.5	3.53	4
3	15.5	19.01	19	9.97	17.15	19	4.58	6.55	8	4.83	3.92	5
4	19.5	20.97	24	14.15	18.66	20	7.33	7.75	8	5	4.51	5
5	23	23.22	28	18.4	19.78	24	10	9.37	10	5	5	5
6	26	24.68	30	19.99	21.01	24	13	10.43	11	6	6	6
7	29	25.84	31	22.96	22.7	26	13	10.52	12	-	-	-
8	29.5	27.09	31	24.79	24.02	31	13	11.06	12	-	-	-
9	34	27.76	33	26.08	24.51	32	14	11.71	14	-	-	-
10	36	28.49	34	27.08	24.81	33	14	12.02	14	-	-	-
11	36	29.12	35	28.58	25.49	33	14	12.47	14	-	-	-
12	37	29.59	36	30.33	26.69	34	14	13.19	14	-	-	-
13	38	30.44	37	33.5	28.07	35	14	13.52	14	-	-	-
14	39	31.04	39	35.5	29.28	35	14	14	14	-	-	-
15	39	31.27	39	36	30.03	36	-	-	-	-	-	-
16	39	31.66	39	37	30.37	36	-	-	-	-	-	-
Total	39	39	39	42	42	42	-	-	-	-	-	-

Table 9: Expected number of families that are assigned apartments with rank up to $k = 1, \ldots, 16$ by the Rawlsian, PS and MTAV rules.

	C_{17}			C_{18}			C_{19}			C_{20}		
Position	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV
1	3.5	4.62	5	5.33	8.39	9	2.8	3.14	4	2.21	5.34	5
2	8	6.75	7	7	8.58	9	3.8	3.97	5	5.92	6.74	8
3	9	7.68	9	9.5	9.09	9	3.8	4.34	5	7.17	8.66	9
4	9	8.6	9	9.5	9.15	9	4.83	4.57	5	10.42	10.57	12
5	9	8.8	9	10.25	9.69	10	5.75	5.33	5	14.75	12.5	14
6	9	8.83	9	10.5	10.17	10	7	6.33	7	16	13.53	16
7	9	8.91	9	14	12.18	14	7.89	7.11	7	16	13.84	16
8	9	8.98	9	15	13.09	15	8	8	8	17	14.64	16
9	9	9	9	15	13.27	15	9	9	9	19	15.61	19
10	-	-	-	15	13.65	15	-	-	-	20	16.11	20
11	-	-	-	15	14.19	15	-	-	-	20	16.53	20
12	-	-	-	15	14.27	15	-	-	-	20	17.15	20
13	-	-	-	15	14.61	15	-	-	-	20	17.69	20
14	-	-	-	15	14.9	15	-	-	-	20	17.95	20
15	-	-	-	15	15	15	-	-	-	20	18.61	20
16	-	-	-	-	-	-	-	-	-	20	18.89	20
Total	-	-	-	-	-	-	-	-	-	20	20	20

Table 10: Expected number of families that are assigned apartments with rank up to $k = 1, \ldots, 16$ by the Rawlsian, PS and MTAV rules.

	C_{21}			C_{22}			C_{23}			C_{24}		
Position	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV	Rawls	PS	MTAV
1	8.79	12.44	12	4	3.75	4	5.5	12.16	10	2.38	2.83	3
2	10.71	13.01	14	7	5.38	7	11.31	17.44	15	4.55	3.4	5
3	13.83	14.38	17	7	6.67	7	15.94	20.44	19	4.8	3.97	5
4	15.17	15.24	17	7	6.83	7	24.42	23.16	27	4.8	4.29	5
5	22	18.45	21	7	6.88	7	28.47	25.91	29	5	5	5
6	23	19.15	22	7	6.92	7	32.36	27.93	34	7	6.29	7
7	24	19.79	24	7	7	7	34.92	29.24	35	8	7.46	8
8	24	20.24	24	-	-	-	37.5	30.17	37	8	8	8
9	24	20.51	24	-	-	-	38	31.08	37	-	-	-
10	24	20.88	24	-	-	-	38.5	31.54	38	-	-	-
11	24	21.69	24	-	-	-	40	32.24	40	-	-	-
12	24	21.84	24	-	-	-	40	32.63	40	-	-	-
13	24	22.2	24	-	-	-	40	32.99	40	-	-	-
14	24	22.35	24	-	-	-	40	33.68	40	-	-	-
15	24	22.67	24	-	-	-	40	33.99	40	-	-	-
16	24	22.77	24	-	-	-	40	34.35	40	-	-	-
Total	24	24	24	-	-	-	40	40	40	-	-	-