THE FUNDAMENTAL SOLUTION OF THE FRACTIONAL $p-$LAPLACIAN

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ABSTRACT. In this article, we find the fundamental solution of the fractional $p-$laplacian and use them to prove two different Liouville–type theorems. A non-existence classical Liouville-type theorem for p-superharmonic and a Louville type results for an Emden-Folder type equation with the fractional $p-$laplacian.

Key words and phrases Fundamental solution; Liouville-type theorems; fractional $p-$laplacian; non-local operator.

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1. Introduction

In the last decades, Liouville-type non-existence theorems have been studied intensively, because they have emerged as a crucial tool for many applications in PDEs. They mostly appear in establishing qualitative properties of solutions. The best known is the Gidas-Spruck a priori bound and nowadays Liouville-type theorems are used in regularity issues. Observe that the non-existence results are used, in most of the cases, after rescaling and a compactness argument. See, for instance, [32, 31, 17, 6] and references therein.

Our purpose here is to establish Liouville-type theorems for equations that involve the fractional p-laplacian, defined by,

$$(-\Delta_p)^s u(x) := 2C(s,p,N) \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{\Psi_p(u(x)-u(y))}{|x-y|^{N+sp}} dy \quad x \in \mathbb{R}^N$$

where $p \in (1,\infty)$, $s \in (0,1)$, and $C(s,p,N)$ is a normalization factor. This constant ensures that in the limits $p \to 2$ and $s \to 1$, the operator coincides with the standard fractional laplacian and the classical $p-$laplacian, respectively. In this work, for simplicity, we omit this constant, as it does not affect the qualitative properties of the solutions we study.

The non-local operators have become relevant because they arise in several applications in many fields, for instance, game theory, mathematical physics, finance, image processing, Lévy processes in probability, and some optimization problems, see [11, 12, 21, 5, 16] and the references therein. From a mathematical point of view, the fractional $p-$Laplacian has a great interest since it exhibits two key features: the nonlinearity of the operator and its non-local character. For instance, the literature includes works on global bifurcation [13], eigenvalues [9, 10, 27], regularity [15, 24, 25], evolution problems [29, 33], and existence results via Moser iteration [23], each contributing key insights into different aspects of equations involving the fractional $p-$laplacian.

1.1. **Main results.**

Our first result introduces the general formula for a fundamental solution of the fractional $p-$laplacian.

Theorem 1.1. *Let $N \geq 2$, $0 < s < 1$, and $1 < p < \infty$.*

(a) If $ps \neq N$ then

$$v_\beta(x) = |x|^\beta \quad \beta \in \left(-\tfrac{N}{p-1}, \tfrac{ps}{p-1}\right),$$

is a weak solution of

$$(-\Delta_p)^s v_\beta(x) = \mathcal{C}(\beta)|x|^{\beta(p-1)-sp} \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where

$$\mathcal{C}(\beta) := 4\pi\alpha_N \int_0^1 \left|1-\rho^\beta\right|^{p-2} \left(1-\rho^\beta\right) \left[\rho^{N-1} - \rho^{ps-\beta(p-1)-1}\right] G(\rho^2)d\rho, \tag{1}$$

with

$$\alpha_N := \frac{\pi^{\frac{N-3}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \text{ and}$$

$$G(t) := G(t, N, ps) := B\left(\frac{N-1}{2}, \frac{1}{2}\right) F\left(\frac{N+ps}{2}, \frac{ps+2}{2}; \frac{N}{2}; t\right).$$

Here Γ, B and F denote the gamma, the beta, and the (2-1)-hypergeometric functions, respectively.

Additionally, we have

$$\mathcal{C}(\beta) \begin{cases} = 0 & \text{if } \beta = 0, \text{ or } \beta = \frac{ps-N}{p-1}, \\ > 0 & \text{if } \min\{\frac{ps-N}{p-1}, 0\} < \beta < \max\{\frac{ps-N}{p-1}, 0\}, \\ < 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) If $ps = N$ then

$$v(x) = \log(|x|),$$

is a weak solution of

$$(-\Delta_p)^s v(x) = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

It should be noted that point *(a)* of the above theorem is mentioned in [7, Example 1.5], but this is only in the case where $2 < p \leq N+1$ and $0 < s < \frac{p-1}{p}$, and without a detailed proof. Additionally, in [8, Appendix A], the result is proven for the case $N > sp$ and $\frac{N-sp}{p} \leq \beta < \frac{N}{p-1}$. Here, we provide a complete proof covering all cases, which requires delicate estimates along with some nontrivial explicit computations.

We also observe that a similar result holds for $N = 1$, though in this work we focus on the case $N \geq 2$.

The assertion of Theorem 1.2 is formally natural due to the scaling properties of the fractional integral operator. Indeed, when considering a rescaling, $u_\lambda(x) = u(\lambda x)$, the fractional integral scales accordingly, suggesting that the result should hold at least in a heuristic sense. However, making this argument rigorous presents substantial challenges due to regularity issues arising from the nonlocal nature of the operator. Establishing the result requires careful analysis, particularly in the delicate logarithmic case. Moreover, determining the precise sign of the constant $\mathcal{C}(\beta)$ in terms of β is itself a subtle and nontrivial task.

Remark 1.1. *By [3], v_β and v solutions are viscosity solutions. Furthermore, since our solutions are $C^\infty(\mathbb{R}^N \setminus \{0\})$ viscosity solutions that have non-zero gradients in $\mathbb{R}^N \setminus \{0\}$, they are also classical solutions. For the definitions of viscosity and weak solutions, see Section 2.*

Remark 1.2. *A review of fundamental solutions in the local case even for fully nonlinear operators can be found in [2]. In this scenario, distributional setting is not possible so the equations hold only in $\mathbb{R}^N\setminus\{0\}$, this is our approach. In the classical literature fundamental solution satisfies also the equation in $\mathbb{R}^N$ with δ_0 as a right hand side. In our setting when $ps-N\neq 0$ the fundamental solution corresponds to the case $\beta=\frac{ps-N}{p-1}$. for the above power function and in the standard case (that is $p=2$ and $s=1$), we have $\beta=2-N$. Here, we abuse of the fundamental solution name since we include all values of the power β in the corresponding admissible range.*

In a parabolic context, the fundamental solution for the fractional $p-$laplacian in self-similar variables is found in [34].

As mentioned before, our main application of the fundamental solution of the fractional $p-$laplacian is to establish two models of Liouville-type theorems depending on the order between N and ps. We start with the case $N\leq ps$ that corresponds to the classical type of Liouville results.

Theorem 1.2 (First Liouville-type theorem)**.** *Let $N\geq 2, 0<s<1$, and $1<p<\infty$. If $N\leq ps$ and u is a non-negative lower semi-continuous weak solution of*

$$(-\Delta_p)^s u\geq 0 \quad \text{in } \mathbb{R}^N, \tag{3}$$

then u is constant.

Remark 1.3. *Again as the previous remark by [26], we know that a non-negative lower semi-continuous weak solution of (3) is also a viscosity solution. Then the previous theorem also holds if we consider viscosity solutions instead of weak ones. See also [3].*

Our second Liouville-type theorem is for an equation involving the fractional $p-$laplacian operator (with $N>ps$) and a zero-order power nonlinearity, this corresponds to in the literature as the Lane-Emden type equations.

Theorem 1.3 (Second Liouville-type theorem)**.** *Let $N\geq 2, 0<s<1$, $1<p<\infty$, and $N>ps$.*

- *If $0<q<\frac{N(p-1)}{N-ps}$ and $u\in C(\mathbb{R}^N)$ is a non-negative viscosity solution of*

$$(-\Delta_p)^s u-u^q\geq 0 \quad \text{in } \mathbb{R}^N \tag{4}$$

  then $u\equiv 0$.
- *If $q>\frac{N(p-1)}{N-ps}$ then there is a positive solution of (4).*

For the proofs of our Liouville-type theorems, we proceed similarly to [17], but we need some extra delicate estimates together with some new ideas due to the strongly nonlinear character of the operator. A review on Liouville-type theorems of this type can be found in [17], for other previous results see also [1, 4, 20, 30].

For the p-laplacian (case $s=1$), the last two theorems were proved in [6], see also [30] where even systems are consider. Furthermore, in [6], it is shown that the first point of Theorem 1.3 also holds when $q=\frac{N(p-1)}{N-ps}$, sometimes known as the critical case for super-solution. Unfortunately, we have not yet been able to establish this result within our current framework, so we leave it as an open problem, stated below. The difficulty in our approach is to compute a log perturbation of the fundamental solution thus is related with a product rule for the fractional $p-$laplacian.

Open problem: *Let $N\geq 2, 0<s<1$, $1<p<\infty$, and $N>ps$. If $q=\frac{N(p-1)}{N-p}$ and $u\in C(\mathbb{R}^N)$ is a non-negative viscosity solution of (4) then $u\equiv 0$.*

In this work, we focus on dimensions $N \geq 2$ and exclude the one-dimensional case $N = 1$. We note, however, that the case $N = 1$ might offer additional insight into the behavior of solutions, potentially simplifying certain arguments or even allowing the construction of explicit counterexamples to open questions, such as higher regularity. This case could be an interesting direction for future investigation.

The paper is organized as follows. In Section 2, we give the definition of weak and viscosity solutions. In Section 3, we prove Theorem 1.1. Afterward, in Section 4, we prove some Hadamard properties that will be fundamental to proving our Liouville results. Finally, in Sections 5 and 6, we prove Theorems 1.2 and 1.3.

2. PRELIMINARIES

Throughout this paper, Ω is an open set of $\mathbb{R}^N$, and $s \in (0,1)$, $p \in (1,\infty)$. The fractional Sobolev spaces $W^{s,p}(\Omega)$ is defined to be the set of functions $u \in L^p(\Omega)$ such that

$$|u|^p_{W^{s,p}(\Omega)} := \int_{\Omega^2} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dxdy < \infty,$$

where Ω^2 denotes $\Omega \times \Omega$. The fractional Sobolev spaces admit the following norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + |u|^p_{W^{s,p}(\Omega)} \right)^{\frac{1}{p}},$$

where

$$\|u\|^p_{L^p(\Omega)} := \int_\Omega |u(x)|^p\, dx.$$

We also denote

$$L^{p-1}_s(\mathbb{R}^N) := \left\{ u \in L^{p-1}_{loc}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} \frac{|u|^{p-1}}{(1+|x|)^{N+sp}} dx < \infty \right\}.$$

For convenience, we will adopt the notation $\Psi_p(t) = |t|^{p-2}t$ throughout this paper.

Remark 2.1. *Let us note that, in Theorem 1.1, we chose $\beta \in \left(-\frac{N}{p-1}, \frac{ps}{p-1}\right)$ so that $v_\beta(x) = |x|^\beta \in L^{p-1}_s(\mathbb{R}^N)$.*

Definition 2.1 (Weak solution)**.** *Let $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function. A function $u \in L^{p-1}_s(\mathbb{R}^N)$ is a weak super-solution (sub-solution) of*

$$(-\Delta_p)^s u(x) = f(x,u) \text{ in } \Omega, \tag{5}$$

if for any bounded open $U \subseteq \Omega$ we have that $u \in W^{s,p}_{loc}(U)$ and

$$\int_{\mathbb{R}^{2N}} \frac{\Psi_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} dxdy \geq (\leq) \int_{\mathbb{R}^N} f(x,u)\varphi(x)dx,$$

for any non-negative function $\varphi \in C^\infty_c(U)$. We say that u is a weak solution of (5) if it is both a weak super-solution and sub-solution to the problem.

Following [26], we define our notion of viscosity super-solution of (5). We start to introduce some notation.

The set of critical points of a differentiable function u and the distance from the critical points are denoted by

$$N_u := \{x \in \Omega \colon \nabla u(x) = 0\}, \quad \text{and} \quad d_u(x) := \operatorname{dist}(x, N_u),$$

respectively. Let $D \subset \Omega$ be an open set. We denote the class of C^2-functions whose gradient and Hessian are controlled by d_u as

$$C^2_\gamma(D) := \left\{ u \in C^2(\Omega) \colon \sup_{x\in D} \left(\frac{\min\{d_u(x),1\}^{\gamma-1}}{|\nabla u(x)|} + \frac{|D^2u(x)|}{d_u(x)^{\gamma-2}} \right) < \infty \right\}.$$

Observe that, if $\gamma \geq 2$ then $u(x) = |x|^\gamma \in C^2_\gamma$.

Definition 2.2 (Viscosity Solution)**.** *Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We say that a function $u\colon \mathbb{R}^N \to [-\infty, \infty]$ is a viscosity super-solution (sub-solution) of (5) if it satisfies the following four assumptions:*

(VS1) $u < \infty$ ($u > -\infty$) a.e. in $\mathbb{R}^N$ and $u > -\infty$ ($u < \infty$) everywhere in Ω;
(VS2) u is lower (upper) semi-continuous in Ω;
(VS3) If $\phi \in C^2(B_r(x_0))$ for some $B_r(x_0) \subset \Omega$ such that $\phi(x_0) = u(x_0)$ and $\phi \leq u$ ($\phi \geq u$) in $B_r(x_0)$, and one of the following holds
 (a) $p > \frac{2}{2-s}$ or $\nabla\phi(x_0) \neq 0$;
 (b) $1 < p \leq \frac{2}{2-s}$; $\nabla\phi(x_0) = 0$ such that x_0 is an isolate critical point of ϕ, and $\phi \in C^2_\gamma(B_r(x_0))$ for some $\gamma > \frac{sp}{p-1}$;
 then $(-\Delta_p)^s\phi_r(x_0) \geq (\leq) f(x_0, u)$, where

$$\phi_r(x) = \begin{cases} \phi & \text{if } x \in B_r(x_0), \\ u(x) & \text{otherwise}; \end{cases}$$

(VS4) $u_- \coloneqq \max\{-u, 0\}$ ($u_+ \coloneqq \max\{u, 0\}$) belongs to $\in L^{p-1}_s(\mathbb{R}^N)$.

Finally, u is a viscosity solution if it is both a viscosity super-solution and sub-solutions.

3. Fundamental solution

In this section, we will prove the main result of this article (Theorem 1.1). To simplify the presentation, we split the proof into two cases.

3.1. Case $ps \neq N$.

This subsection aims to prove the following result.

Theorem 3.1. *Let $N \geq 2, 0 < s < 1$, and $1 < p < \infty$. If $ps \neq N$ then*

$$v_\beta(x) = |x|^\beta \quad \beta \in \left(-\tfrac{N}{p-1}, \tfrac{ps}{p-1}\right),$$

is a weak solution of

$$(-\Delta_p)^s v_\beta(x) = \mathcal{C}(\beta)|x|^{\beta(p-1)-sp} \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{6}$$

where $\mathcal{C}(\beta)$ is defined by (1).

We introduce a bit of notation to take advantage of the fact that v_β is a radial function. Given $\varepsilon > 0$ and we define

$$A_\varepsilon(x) \coloneqq \{y \in \mathbb{R}^N : ||x| - |y|| < \varepsilon\} = B_{|x|+\varepsilon}(0) \setminus \overline{B_{|x|-\varepsilon}}(0)$$

and

$$J_\varepsilon v_\beta(x) \coloneqq 2 \int\limits_{\mathbb{R}^N\setminus A_\varepsilon(x)} \frac{\Psi_p(v_\beta(x) - v_\beta(y))}{|x-y|^{N+sp}} dy.$$

Notice that $J_\varepsilon v_\beta(x)$ is finite for all $x \in \mathbb{R}^N$ since $\beta \in \left(-\frac{N}{p-1}, \frac{ps}{p-1}\right)$.

Ideas of the proof of Theorem 3.1. We proceed somewhat as in the proof of [19, Proof of Lemma 3.1].

First, we will show that

$$J_\varepsilon v_\beta(x) = h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp} - g_{\beta,\varepsilon}(x) \text{ in } \mathbb{R}^N \setminus \{0\}, \tag{7}$$

where

$$h_{\beta,\varepsilon}(x) \coloneqq \pi\alpha_N \int\limits_0^{1-\frac{\varepsilon}{|x|}} \Psi_p(1-\rho^\beta)\left[\rho^{N-1} - \rho^{ps-\beta(p-1)-1}\right] G(\rho^2, N, ps)d\rho,$$

$$g_{\beta,\varepsilon}(x) \coloneqq 4\pi\alpha_N |x|^{\beta(p-1)-sp} \int\limits_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \Psi_p(1-\rho^\beta)\rho^{sp-\beta(p-1)-1} G(\rho^2, N, ps)d\rho.$$

The function G is defined in Theorem 1.1.

Then we will show that

$$h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp} \to \mathcal{C}(\beta)|x|^{\beta(p-1)-sp} \quad \text{and} \quad g_{\beta,\varepsilon}(x) \to 0,$$

strongly in $L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ as $\varepsilon \to 0^+$.

Finally, we will conclude our result.

Proof of Theorem 3.1. Throughout this proof, we assume that $ps \neq N$, $\beta \in \left(-\frac{N}{p-1}, \frac{ps}{p-1}\right)$, $\varepsilon > 0$ and $x \in \mathbb{R}^{\mathbb{N}} \setminus \{0\}$, and we simply use the notations $G(t)$ instead of $G(t, N, ps)$. We split the proof into four steps.

Step 1. The first part of the proof shows (7).

We begin by observing that

$$\begin{aligned} J_\varepsilon v_\beta(x) &= 2 \int\limits_{\mathbb{R}^N \setminus A_\varepsilon(x)} \frac{\Psi_p(v_\beta(x) - v_\beta(y))}{|x-y|^{N+sp}} dy \\ &= 2 \int\limits_{\mathbb{R}^N \setminus A_\varepsilon(x)} \frac{||x|^\beta - |y|^\beta|^{p-2}(|x|^\beta - |y|^\beta)}{|x-y|^{N+sp}} dy \\ &= 2|x|^{\beta(p-1)-sp} \int\limits_{\mathbb{R}^N \setminus A_\varepsilon(x)} \frac{\left|1 - \left(\frac{|y|}{|x|}\right)^\beta\right|^{p-2}\left(1 - \left(\frac{|y|}{|x|}\right)^\beta\right)}{\left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{N+sp}} \frac{dy}{|x|^N}. \end{aligned}$$

Thus, by a simple change of variable and by rotation invariance, we have that

$$J_\varepsilon v_\beta(x) = 2|x|^{\beta(p-1)-sp} \int\limits_{\mathbb{R}^N \setminus A_{\varepsilon/|x|}(e_1)} \frac{\Psi_p\left(1 - |y|^\beta\right)}{|e_1 - y|^{N+sp}} dy,$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$.

We now take $y = \rho z$ with $\rho > 0$ and $z \in \mathbb{S}^{N-1} \coloneqq \{w \in \mathbb{R}^N : |w| = 1\}$, we get

$$\begin{aligned} J_\varepsilon v_\beta(x) &= 2|x|^{\beta(p-1)-sp} \int\limits_{|1-\rho| \geq \frac{\varepsilon}{|x|}} \Psi_p\left(1-\rho^\beta\right) \int\limits_{\mathbb{S}^{N-1}} \frac{d\mathcal{H}^{N-1}(z)}{|e_1 - \rho z|^{N+sp}} \rho^{N-1} d\rho \\ &= 2|x|^{\beta(p-1)-sp} \int\limits_{|1-\rho| \geq \frac{\varepsilon}{|x|}} \Psi_p\left(1-\rho^\beta\right) \int\limits_{\mathbb{S}^{N-1}} \frac{d\mathcal{H}^{N-1}(z)}{|1 - 2\rho e_1 \cdot z + \rho^2|^{\frac{N+sp}{2}}} \rho^{N-1} d\rho. \end{aligned}$$

Using [18, page 249] and [22, 3.665 (427)], we have that

$$J_\varepsilon v_\beta(x) = 4\pi\alpha_N |x|^{\beta(p-1)-sp} \int\limits_{|1-\rho| \geq \frac{\varepsilon}{|x|}} \Psi_p\left(1-\rho^\beta\right) \rho^{N-1} \mathcal{K}(\rho) d\rho,$$

where

$$\mathcal{K}(\rho) := \int_0^\pi \frac{\sin^{N-2}(\theta)d\theta}{|1-2\rho\cos(\theta)+\rho^2|^{\frac{N+sp}{2}}} = \begin{cases} G(\rho^2) & \text{if } \rho < 1, \\ \frac{G(\rho^{-2})}{\rho^{N+ps}} & \text{if } \rho > 1. \end{cases} \tag{8}$$

Therefore,

$$\begin{aligned} J_\varepsilon v_\beta(x) &= 4\pi\alpha_N |x|^{\beta(p-1)-sp} \Bigg\{ \int_{1+\frac{\varepsilon}{|x|}}^{\infty} \Psi_p\left(1-\rho^\beta\right) \frac{G(\rho^{-2})}{\rho^{ps+1}} d\rho \\ &\qquad + \int_0^{1-\frac{\varepsilon}{|x|}} \Psi_p\left(1-\rho^\beta\right)\rho^{N-1}G(\rho^2)d\rho \Bigg\} \\ &= 4\pi\alpha_N |x|^{\beta(p-1)-sp} \Bigg\{ \int_0^{\frac{|x|}{|x|+\varepsilon}} \Psi_p\left(1-\rho^{-\beta}\right)\rho^{ps-1}G(\rho^2)d\rho \\ &\qquad + \int_0^{1-\frac{\varepsilon}{|x|}} \Psi_p\left(1-\rho^\beta\right)\rho^{N-1}G(\rho^2)d\rho \Bigg\} \\ &= 4\pi\alpha_N |x|^{\beta(p-1)-sp} \Bigg\{ \int_0^{1-\frac{\varepsilon}{|x|}} \Psi_p\left(1-\rho^\beta\right)\left[\rho^{N-1}-\rho^{ps-\beta(p-1)-1}\right]G(\rho^2)d\rho \\ &\qquad - \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \Psi_p\left(1-\rho^\beta\right)\rho^{ps-\beta(p-1)-1}G(\rho^2)d\rho \Bigg\}. \end{aligned}$$

So,

$$J_\varepsilon v_\beta(x) = h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp} - g_{\beta,\varepsilon}(x) \text{ in } \mathbb{R}^N\setminus\{0\}.$$

Step 2. We now show that

$$h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp} \to \mathcal{C}(\beta)|x|^{\beta(p-1)-sp} \quad \text{strongly in } L^1_{\text{loc}}(\mathbb{R}^N\setminus\{0\}),$$

as $\varepsilon \to 0^+$.

Given a bounded set $U \subset \mathbb{R}^N \setminus \{0\}$ such that $\overline{U} \subset \mathbb{R}^N \setminus \{0\}$, we want to show that

$$\begin{aligned} 0 &= \frac{1}{4\pi\alpha_N} \lim_{\varepsilon\to0^+} \int_U |h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp} - \mathcal{C}(\beta)|x|^{\beta(p-1)-sp}|dx \\ &= \frac{1}{4\pi\alpha_N} \lim_{\varepsilon\to0^+} \int_U |h_{\beta,\varepsilon}(x) - \mathcal{C}(\beta)|\,|x|^{\beta(p-1)-sp}dx \\ &= \lim_{\varepsilon\to0^+} \int_U |x|^{\beta(p-1)-sp} \left| \int_{1-\frac{\varepsilon}{|x|}}^{1} \Psi_p\left(1-\rho^\beta\right)(\rho^{N-1}-\rho^{ps-\beta(p-1)-1})G(\rho^2)d\rho \right| dx. \end{aligned}$$

Since $\overline{U} \subset \mathbb{R}^N \setminus \{0\}$ is bounded, we have that $|x|^{\beta(p-1)-sp} \in L^\infty(U)$. Therefore, it is enough to prove that

$$\lim_{\varepsilon\to 0^+} \int_U \int_{1-\frac{\varepsilon}{|x|}}^{1} |1-\rho^\beta|^{p-1} |\rho^{N-1} - \rho^{ps-\beta(p-1)-1}| G(\rho^2) d\rho dx = 0. \tag{9}$$

Let $H(\rho) \coloneqq (1-\rho)^{1+ps} G(\rho^2)$. By [28, page 271, (2)], we have that

$$\lim_{\rho\to 1^-} H(\rho) \tag{10}$$

exists. Then,

$$|1-\rho^\beta|^{p-1}|\rho^{N-1} - \rho^{ps-\beta(p-1)-1}|G(\rho^2) = \frac{|1-\rho^\beta|^{p-1}|\rho^{N-1} - \rho^{ps-\beta(p-1)-1}|}{|1-\rho|^{1+ps}} H(\rho),$$

belongs to $L^1(0,1)$. Therefore, by the dominated convergence theorem, we get (9).

Step 3. Our next goal is to show that,

$$g_{\beta,\varepsilon}(x) \to 0 \quad \text{strongly in } L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}). \tag{11}$$

as $\varepsilon \to 0^+$.

Again, let $U \subset \mathbb{R}^N \setminus \{0\}$ be a bounded set such that $\overline{U} \subset \mathbb{R}^N \setminus \{0\}$. In this case, we want to show that

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0^+} \int_U |g_{\beta,\varepsilon}(x)| dx \\ &= 4\pi\alpha_N \lim_{\varepsilon\to 0^+} \int_U |x|^{\beta(p-1)-sp} \left| \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \Psi_p\left(1-\rho^\beta\right) \rho^{sp-\beta(p-1)-1} G(\rho^2) d\rho \right| dx. \end{aligned}$$

Since $\overline{U} \subset \mathbb{R}^N \setminus \{0\}$ is bounded, we have that $|x|^{\beta(p-1)-sp} \in L^\infty(U)$, and

$$\left|1-\rho^\beta\right| \le C|1-\rho|,$$

where C is a positive constant that depends on β, p, and $\text{dist}(U, 0)$. Then

$$\int_U |g_{\beta,\varepsilon}(x)| dx \le C \int_U \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} |1-\rho|^{p-1} G(\rho^2) d\rho dx = C \int_U \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{|1-\rho|^{p(s-1)-2}} d\rho dx,$$

where C is a positive constant that depends on $\||x|^{\beta(p-1)-sp}\|_{L^\infty(U)}$, β, p, and $\text{dist}(U, 0)$. Therefore, to show (11), it is enough to show that

$$\lim_{\varepsilon\to 0^+} \int_U \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{|1-\rho|^{p(s-1)+2}} d\rho dx = 0. \tag{12}$$

To prove this, we consider three cases.

Case 1: $p > \frac{1}{1-s}$.

By (10), $\dfrac{H(\rho)}{|1-\rho|^{p(s-1)+2}} \in L^1(0,1)$. Thus, by the dominated convergence theorem, we get (12).

Case 2: We now assume $p < \frac{1}{1-s}$.

By [28, pages 257 (5) and 271 (2)], we have that H is differentiable and

$$\lim_{\rho\to 1^-} H'(\rho) \tag{13}$$

exists. Notice that for any $x \in U$ we have

$$\int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{|1-\rho|^{p(s-1)+2}} d\rho =$$

$$\frac{1}{p(s-1)+1} \left\{ \frac{H\left(\frac{|x|}{|x|+\varepsilon}\right)}{\left(\frac{\varepsilon}{|x|+\varepsilon}\right)^{p(s-1)+1}} - \frac{H\left(1-\frac{\varepsilon}{|x|}\right)}{\left(\frac{\varepsilon}{|x|}\right)^{p(s-1)+1}} - \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H'(\rho)}{|1-\rho|^{p(s-1)+1}} d\rho \right\}.$$

Taking $\delta = \mathrm{dist}(U, 0)$, and using (10) and (13), there are two positive constants C_1 and C_2 such for any $x \in U$ we have

$$\left| \frac{H\left(\frac{|x|}{|x|+\varepsilon}\right)}{\left(\frac{\varepsilon}{|x|+\varepsilon}\right)^{p(s-1)+1}} - \frac{H\left(1-\frac{\varepsilon}{|x|}\right)}{\left(\frac{\varepsilon}{|x|}\right)^{p(s-1)+1}} \right| =$$

$$\begin{aligned}
&\leq \varepsilon^{p(1-s)} (|x|+\varepsilon)^{p(s-1)+1} \frac{\left| H\left(\frac{|x|}{|x|+\varepsilon}\right) - H\left(1-\frac{\varepsilon}{|x|}\right) \right|}{\varepsilon} \\
&\quad + H\left(1-\frac{\varepsilon}{|x|}\right) \frac{\left| (|x|+\varepsilon)^{p(s-1)+1} - |x|^{p(s-1)+1} \right|}{\varepsilon} \\
&\leq \varepsilon^{p(1-s)} \left\{ \frac{\|H'\|_{L^\infty(0,1)}}{\delta^{p(1-s)+1}} + \frac{p(1-s)}{\delta^{p(s-1)}} \|H\|_{L^\infty(0,1)} \right\} \\
&\leq C_1 \varepsilon^{p(1-s)},
\end{aligned}$$

and

$$\int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H'(\rho)}{|1-\rho|^{p(s-1)+1}} d\rho \leq C_2 \varepsilon^{p(1-s)}.$$

Then

$$\int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{|1-\rho|^{p(s-1)+2}} d\rho \leq C_1 \varepsilon^{p(1-s)} + C_2 \varepsilon^{p(1-s)} \quad \forall x \in U.$$

This implies (12).

Case 3: Finally we consider the case $p = \frac{1}{1-s}$.

In this case, for any $x \in U$ we have

$$\int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{|1-\rho|^{p(s-1)+2}} d\rho = \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{|1-\rho|} d\rho$$

$$= H\left(\frac{|x|}{|x|+\varepsilon}\right) \log\left(\frac{\varepsilon}{|x|+\varepsilon}\right) - H\left(1-\frac{\varepsilon}{|x|}\right) \log\left(\frac{\varepsilon}{|x|}\right) - \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} H'(\rho)\log(1-\rho) d\rho$$

$$= \left\{ H\left(\frac{|x|}{|x|+\varepsilon}\right) - H\left(1-\frac{\varepsilon}{|x|}\right)\right\} \log\left(\frac{\varepsilon}{|x|+\varepsilon}\right)$$

$$+ H\left(1-\frac{\varepsilon}{|x|}\right) \left\{ \log\left(\frac{\varepsilon}{|x|+\varepsilon}\right) - \log\left(\frac{\varepsilon}{|x|}\right)\right\} - \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} H'(\rho)\log(1-\rho) d\rho$$

$$= \left\{ H\left(\frac{|x|}{|x|+\varepsilon}\right) - H\left(1-\frac{\varepsilon}{|x|}\right)\right\} \log\left(\frac{\varepsilon}{|x|+\varepsilon}\right) + H\left(1-\frac{\varepsilon}{|x|}\right) \log\left(\frac{|x|}{|x|+\varepsilon}\right)$$

$$- \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} H'(\rho)\log(1-\rho) d\rho.$$

Again, taking $\delta = \operatorname{dist}(U, 0)$, and using (10) and (13), there are three positive constants C_1, C_2 and C_3 such for any $x \in U$ we have

$$\int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{|1-\rho|^{p(s-1)+2}} d\rho \le$$

$$\le -\frac{\|H'\|_{L^\infty(0,1)}}{\delta^2}\varepsilon^2 \log\left(\frac{\varepsilon}{\delta+\varepsilon}\right) + \|H\|_{L^\infty(0,1)}\frac{\varepsilon}{\delta} - \|H'\|_{L^\infty(0,1)} \int_{1-\frac{\varepsilon}{\delta}}^{1} \log(1-\rho) d\rho$$

$$\le -\frac{\|H'\|_{L^\infty(0,1)}}{\delta^2}\varepsilon^2 \log\left(\frac{\varepsilon}{\delta+\varepsilon}\right) + \|H\|_{L^\infty(0,1)}\frac{\varepsilon}{\delta} - \|H'\|_{L^\infty(0,1)}\frac{\varepsilon}{\delta}\left(\log(\varepsilon) - \log(\delta) - 1\right)$$

$$\le -C_1\varepsilon^2 \log\left(\frac{\varepsilon}{\delta+\varepsilon}\right) - C_2\varepsilon\log(\varepsilon) + C_3\varepsilon,$$

from which (12) follows.

Step 4. Finally, we will show that v_β is a weak solution of (6).

By step 1, we have that

$$J_\varepsilon v_\beta(x) = h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp} - g_{\beta,\varepsilon}(x) \text{ in } \mathbb{R}^N \setminus \{0\}.$$

Then, given $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ we have that

$$\int_{\mathbb{R}^N} J_\varepsilon v_\beta(x)\varphi(x) dx = \int_{\mathbb{R}^N} \left(h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp} - g_{\beta,\varepsilon}(x)\right)\varphi(x)\, dx,$$

that is

$$\int_{\mathbb{R}^{2N}} (1-\chi_{A_\varepsilon(x)}(y))\frac{\Psi_p(v_\beta(x)-v_\beta(y))}{|x-y|^{N+sp}}\varphi(x)dydx = \frac{1}{2}\int_{\mathbb{R}^N}\Big(h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp}-g_{\beta,\varepsilon}(x)\Big)\varphi(x)\,dx. \tag{14}$$

Interchanging the roles of x and y

$$\int_{\mathbb{R}^{2N}} (1-\chi_{A_\varepsilon(y)}(x))\frac{\Psi_p(v_\beta(y)-v_\beta(x))}{|x-y|^{N+sp}}\varphi(y)dxdy = \frac{1}{2}\int_{\mathbb{R}^N}\Big(h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp}-g_{\beta,\varepsilon}(x)\Big)\varphi(x)\,dx \tag{15}$$

Now, adding (14) and (15) and using that $1-\chi_{A_\varepsilon(y)}(x)=1-\chi_{A_\varepsilon(x)}(y)$, we get

$$\int_{\mathbb{R}^N}\Big(h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp}-g_{\beta,\varepsilon}(x))\Big)\varphi(x)\,dx = \int_{\mathbb{R}^{2N}} (1-\chi_{A_\varepsilon(x)}(y))\frac{\Psi_p(v_\beta(x)-v_\beta(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dydx. \tag{16}$$

By steps 2 and 3

$$\int_{\mathbb{R}^N}\Big(h_{\beta,\varepsilon}(x)|x|^{\beta(p-1)-sp}-g_{\beta,\varepsilon}(x))\Big)\varphi(x)\,dx \to \mathcal{C}(\beta)\int_{\mathbb{R}^N}|x|^{\beta(p-1)-sp}\varphi(x)dx. \tag{17}$$

as $\varepsilon\to 0^+$.

On the other hand, we have that

$$\frac{\Psi_p(v_\beta(x)-v_\beta(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \in L^1(\mathbb{R}^N\times\mathbb{R}^N),$$

and

$$\chi_{A_\varepsilon(x)}(y)\to 0 \text{ a.e. in } \mathbb{R}^{2N} \text{ as } \varepsilon\to 0^+.$$

Then

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^{2N}} (1-\chi_{A_\varepsilon(x)}(y))\frac{\Psi_p(v_\beta(x)-v_\beta(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dydx = \int_{\mathbb{R}^{2N}}\frac{\Psi_p(v_\beta(x)-v_\beta(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dydx. \tag{18}$$

Since φ is arbitrary, by (16), (17) and (18), we conclude that v_β is a weak solution of (6). □

3.2. **Case $ps=N$.**

To complete study of the fundamental solution of the fractional $p-$laplacian, we prove the following result.

Theorem 3.2. *Let $N\geq 2, 0<s<1,$ and $1<p<\infty$. If $ps=N$ then*

$$v(x)=\log(|x|),$$

is a weak solution of

$$(-\Delta_p)^s v(x)=0 \quad \text{in } \mathbb{R}^N\setminus\{0\}. \tag{19}$$

Ideas of the proof. Given $\varepsilon > 0$, we define

$$J_\varepsilon v(x) := 2 \int\limits_{\mathbb{R}^N \setminus A_\varepsilon(x)} \frac{\Psi_p(v(x)-v(y))}{|x-y|^{N+sp}} dy.$$

First, we will show that

$$J_\varepsilon v(x) \to 0 \text{ strongly in } L^1_{\text{loc}}(\Omega)$$

as $\varepsilon \to 0^+$. Then, arguing as in step 4 of the proof of Theorem 3.1, we conclude that v is a weak solution of (19).

Proof of Theorem 3.2. Throughout this proof, we assume that $ps = N$, $\varepsilon > 0$ and $x \in \mathbb{R}^{\mathbb{N}} \setminus \{0\}$.

Let us begin by observing that

$$\begin{aligned} J_\varepsilon v(x) &= 2 \int\limits_{\mathbb{R}^N \setminus A_\varepsilon(x)} \frac{\Psi_p(v(x)-v(y))}{|x-y|^{N+sp}} dy \\ &= -\frac{2}{|x|^N} \int\limits_{\mathbb{R}^N \setminus A_\varepsilon(x)} \frac{\left|\log\left(\frac{|y|}{|x|}\right)\right|^{p-2} \log\left(\frac{|y|}{|x|}\right)}{\left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{2N}} \frac{dy}{|x|^N}. \end{aligned}$$

Now, proceeding as in step 1 of the proof of Theorem 3.1

$$J_\varepsilon v(x) = -\frac{4\alpha_N}{|x|^N} \int\limits_{|1-\rho| \geq \frac{\varepsilon}{|x|}} \Psi_p(\log(\rho))\rho^{N-1}\mathcal{K}(\rho)\, d\rho,$$

where $\mathcal{K}(\rho)$ is defined by (8).

Then

$$\begin{aligned} J_\varepsilon v(x) &= -\frac{4\alpha_N}{|x|^N} \left\{ \int\limits_0^{1-\frac{\varepsilon}{|x|}} \Psi_p(\log(\rho))\rho^{N-1} G(\rho^2, N, N)\, d\rho \right. \\ &\qquad \left. + \int\limits_{1+\frac{\varepsilon}{|x|}}^{\infty} \Psi_p(\log(\rho))\rho^{-N-1} G(\rho^{-2}, N, N)\, d\rho \right\} \\ &= -\frac{4\alpha_N}{|x|^N} \left\{ \int\limits_0^{1-\frac{\varepsilon}{|x|}} \Psi_p(\log(\rho))\rho^{N-1} G(\rho^2, N, N)\, d\rho \right. \\ &\qquad \left. - \int\limits_0^{\frac{|x|}{|x|+\varepsilon}} \Psi_p(\log(\rho))\rho^{N-1} G(\rho^2, N, N)\, d\rho \right\} \\ &= \frac{4\alpha_N}{|x|^N} \int\limits_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \Psi_p(\log(\rho))\rho^{N-1} G(\rho^2, N, N)\, d\rho. \end{aligned}$$

That is,

$$J_\varepsilon v(x) = g_\varepsilon(x) \text{ in } \mathbb{R}^N \setminus \{0\} \tag{20}$$

where

$$g_\varepsilon(x) = \frac{4\alpha_N}{|x|^N} \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \Psi_p(\log(\rho))\rho^{N-1} G(\rho^2, N, N)\, d\rho.$$

We claim

$$g_\varepsilon \to 0 \text{ strongly in } L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}), \tag{21}$$

as $\varepsilon \to 0^+$.

To see this, notice that

$$\begin{aligned} |g_\varepsilon(x)| \leq& \frac{4\alpha_N}{|x|(|x|+\varepsilon)^{N-1}} \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{|\log(\rho)|^{p-1}}{(1-\rho)^{N+1}} (1-\rho)^{N+1} G(\rho^2, N, N)\, d\rho \\ \leq& 4\alpha_N \frac{|x|^{p-2}}{(|x|+\varepsilon)^{N-1}(|x|-\varepsilon)^{p-1}} \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{(1-\rho)^{N+1} G(\rho^2, N, N)}{(1-\rho)^{N-p+2}}\, d\rho. \end{aligned} \tag{22}$$

On the other hand, by [28, pages 257 (5) and 271 (2)], we have that $H(\rho) = (1-\rho)^{N+1} G(\rho^2, N, N)$ is differentiable and

$$\lim_{\rho \to 1^-} H(\rho) \text{ and } \lim_{\rho \to 1^-} H'(\rho), \tag{23}$$

exist.

As in step 3 of the proof of Theorem 3.1, we consider three cases to prove our claim.

Case 1: We start assuming that $p > N+1$.

In this case, owing to (22) and (23), it is easy to check that there is a positive constant C (independent of ε)

$$\begin{aligned} |g_\varepsilon(x)| \leq& 4\alpha_N \frac{4\alpha_N |x|^{p-2}}{(|x|+\varepsilon)^{N-1}(|x|-\varepsilon)^{p-1}} \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{(1-\rho)^{N-p+2}}\, d\rho \\ \leq& \frac{4\alpha_N |x|^{p-2}}{(|x|+\varepsilon)^{N-1}(|x|-\varepsilon)^{p-1}} \left(\frac{1}{|x|^{p-N-1}} - \frac{1}{(|x|+\varepsilon)^{p-N-1}} \right) \varepsilon^{p-N-1}. \end{aligned}$$

This implies (21).

Case 2: We now study the case $p < N+1$.

Due to (23), we have that

$$\int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{|1-\rho|^{N-p+2}}d\rho =$$

$$= \frac{1}{N-p+1}\left\{ \frac{H\left(\frac{|x|}{|x|+\varepsilon}\right)}{\left(\frac{\varepsilon}{|x|+\varepsilon}\right)^{N-p+1}} - \frac{H\left(1-\frac{\varepsilon}{|x|}\right)}{\left(\frac{\varepsilon}{|x|}\right)^{N-p+1}} - \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H'(\rho)}{|1-\rho|^{N-p+1}}d\rho \right\}$$

$$\leq \frac{\|H'\|_{L^\infty(0,1)}}{N-p+1}\left\{ \frac{\varepsilon}{(|x|+\varepsilon)^{p-N}|x|} + \frac{1}{p-N}\left(\frac{1}{|x|^{p-N}} - \frac{1}{(|x|+\varepsilon)^{p-N}} \right) \right\}\varepsilon^{p-N}$$

$$+ \frac{\|H\|_{L^\infty(0,1)}}{|x|^{p-N}}\varepsilon^{p-N}.$$

This implies, using again (22),

$$|g_\varepsilon(x)| \leq$$

$$C\varepsilon^{p-N}\frac{|x|^{p-2}}{(|x|+\varepsilon)^{N-1}(|x|-\varepsilon)^{p-1}}\left\{ \frac{\varepsilon}{(|x|+\varepsilon)^{p-N}|x|} + \frac{1}{|x|^{p-N}} - \frac{1}{(|x|+\varepsilon)^{p-N}} \right\}$$

where C is a positive constant independent of ε. Now it is easy to check (21).

Case 3: To conclude the proof of our claim, we study the case $p = N+1$.

Again by (23), we have that

$$\int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{|1-\rho|^{N-p+2}}d\rho = \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} \frac{H(\rho)}{|1-\rho|}d\rho =$$

$$H\left(\frac{|x|}{|x|+\varepsilon}\right)\log\left(\frac{\varepsilon}{|x|+\varepsilon}\right) - H\left(1-\frac{\varepsilon}{|x|}\right)\log\left(\frac{\varepsilon}{|x|}\right) - \int_{1-\frac{\varepsilon}{|x|}}^{\frac{|x|}{|x|+\varepsilon}} H'(\rho)\log(1-\rho)d\rho$$

$$\leq \|H'\|_{L^\infty(0,1)}\left\{ -\frac{\varepsilon\log\left(\frac{\varepsilon}{|x|+\varepsilon}\right)}{(|x|+\varepsilon)|x|} - \left(\frac{\varepsilon\left(\log(\varepsilon)-1\right)}{|x|(|x|+\varepsilon)} + \frac{\log(|x|+\varepsilon)}{(|x|+\varepsilon)} - \frac{\log(|x|)}{|x|} \right) \right\}\varepsilon$$

$$+ \frac{\|H\|_{L^\infty(0,1)}}{|x|}\varepsilon.$$

This implies, using again (22),

$$|g_\varepsilon(x)| \leq$$

$$C\varepsilon|x|^{N-2}\frac{-2\varepsilon\log(\varepsilon) + 2\varepsilon + (\varepsilon - |x|)\log(|x|+\varepsilon) + (|x|+\varepsilon)\log(|x|) + |x|}{(|x|+\varepsilon)^N(|x|-\varepsilon)^N}$$

where C is a positive constant independent of ε. This implies (21).

Finally, we prove that $v(x) = \log(|x|)$, is a weak solution of (19). By (20), we have that

$$J_\varepsilon v(x) = g_\varepsilon(x) \text{ in } \mathbb{R}^N \setminus \{0\}.$$

Then, given $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ we have that

$$\int_{\mathbb{R}^N} J_\varepsilon v(x)\varphi(x)dx = \int_{\mathbb{R}^N} g_\varepsilon(x)\varphi(x)\,dx$$

that is

$$(24)\qquad \int\limits_{\mathbb{R}^{2N}} (1-\chi_{A_\varepsilon(x)}(y))\frac{\Psi_p(v(x)-v(y))}{|x-y|^{N+sp}}\varphi(x)dydx = \frac{1}{2}\int\limits_{\mathbb{R}^N} g_\varepsilon(x)\varphi(x)dx.$$

Interchanging the roles of x and y

$$(25)\qquad \int\limits_{\mathbb{R}^{2N}} (1-\chi_{A_\varepsilon(y)}(x))\frac{\Psi_p(v(y)-v(x))}{|x-y|^{N+sp}}\varphi(y)dxdy = \frac{1}{2}\int\limits_{\mathbb{R}^N} g_\varepsilon(x)\varphi(x)\,dx$$

Now, adding (24) and (25) and using that $1-\chi_{A_\varepsilon(y)}(x) = 1-\chi_{A_\varepsilon(x)}(y)$, we get

$$(26)\qquad \int\limits_{\mathbb{R}^N} g_\varepsilon(x)\varphi(x)\,dx = \int\limits_{\mathbb{R}^{2N}} (1-\chi_{A_\varepsilon(x)}(y))\frac{\Psi_p(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dydx.$$

By (21) we have that

$$(27)\qquad \lim_{\varepsilon\to 0^+}\int\limits_{\mathbb{R}^N} g_\varepsilon(x)\varphi(x)\,dx = 0.$$

On the other hand, we have that

$$\frac{\Psi_p(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \in L^1(\mathbb{R}^{2N})$$

and

$$\chi_{A_\varepsilon(x)}(y)\to 0 \text{ a.e. in } \mathbb{R}^{2N} \text{ as } \varepsilon\to 0^+.$$

Then

$$(28)\qquad \begin{aligned}\lim_{\varepsilon\to 0^+}\int\limits_{\mathbb{R}^{2N}} &(1-\chi_{A_\varepsilon(x)}(y))\frac{\Psi_p(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dydx\\ &= \int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dydx.\end{aligned}$$

Since φ is arbitrary, by (26), (27) and (28), we conclude that v_β is a weak solution of (6). □

4. Hadamard properties

To prove our Hadamard properties, we use comparison techniques that require modifying the fundamental solution near the origin to put it below a weak fractional superharmonic function near the origin.

4.1. **Two fractional subharmonic functions.**

We start by building two weak fractional subharmonic functions from the fundamental solution.

Throughout this section $N\geq 2, 0<s<1, 1<p<\infty, N>ps, \beta\in\left(-\frac{N}{p-1},\frac{ps-N}{p-1}\right)$ and $0<\varepsilon<1<r<R$. Now, define

$$B_\rho := B_\rho(0)\quad (\rho>0),\quad A_{r,R} := \left\{x\in\mathbb{R}^N\colon r<|x|<R\right\},$$

and

$$\phi_\varepsilon(x) := \begin{cases}\varepsilon^\beta & \text{if } 0\leq|x|<\varepsilon,\\ |x|^\beta & \text{if } \varepsilon\leq|x|.\end{cases}$$

Lemma 4.1. *There exists $\varepsilon_0\in(0,1)$ independent of R such that for any $\varepsilon\in(0,\varepsilon_0)$, ϕ_ε is a weak solution of*

$$(-\Delta_p)^s\phi_\varepsilon(x)\leq 0 \text{ in } A_{r,R}.$$

Proof. Let $\varphi \in C_c^\infty(A_{r,R})$ be non-negative.Then

$$\int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(\phi_\varepsilon(x)-\phi_\varepsilon(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy = I_1 + I_2, \tag{29}$$

where

$$\begin{aligned} I_1 &= 2\int\limits_{B_\varepsilon}\int\limits_{\mathbb{R}^N\setminus B_\varepsilon} \frac{\Psi_p(\phi_\varepsilon(x)-\phi_\varepsilon(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy \\ &= -2\int\limits_{\mathbb{R}^N\setminus B_r}\int\limits_{B_\varepsilon} \frac{|\varepsilon^\beta - |x|^\beta|^{p-1}}{|x-y|^{N+sp}}dy\,\varphi(x)dx \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int\limits_{(\mathbb{R}^N\setminus B_\varepsilon)^2} \frac{\Psi_p(\phi_\varepsilon(x)-\phi_\varepsilon(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy \\ &= \int\limits_{(\mathbb{R}^N\setminus B_\varepsilon)^2} \frac{\Psi_p(|x|^\beta-|y|^\beta)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy \\ &= \int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(|x|^\beta-|y|^\beta)(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy \\ &\quad + 2\int\limits_{\mathbb{R}^N\setminus B_r}\int\limits_{B_\varepsilon} \frac{||y|^\beta-|x|^\beta|^{p-1}}{|x-y|^{N+sp}}dy\,\varphi(x)dx. \end{aligned}$$

By Theorem 3.1, we have that

$$I_2 = \mathcal{C}(\beta)\int\limits_{\mathbb{R}^N} |x|^{\beta(p-1)-sp}\varphi(x)dx + 2\int\limits_{\mathbb{R}^N\setminus B_r}\int\limits_{B_\varepsilon} \frac{||y|^\beta-|x|^\beta|^{p-1}}{|x-y|^{N+sp}}dy\,\varphi(x)dx.$$

Then

$$I_1 + I_2 = \mathcal{C}(\beta)\int\limits_{\mathbb{R}^N} |x|^{\beta(p-1)-sp}\varphi(x)dx + 2\int\limits_{\mathbb{R}^N\setminus B_r} F(x)\varphi(x)dx, \tag{30}$$

where

$$F(x) \coloneqq \int\limits_{B_\varepsilon} \frac{||y|^\beta-|x|^\beta|^{p-1} - |\varepsilon^\beta-|x|^\beta|^{p-1}}{|x-y|^{N+sp}}dy = \int\limits_{B_\varepsilon} k(x, \tfrac{y}{|x|})dy\,|x|^{\beta(p-1)-N-sp}$$

with

$$k(x,z) \coloneqq \frac{\left|z^\beta - 1\right|^{p-1} - \left|\left(\frac{\varepsilon}{|x|}\right)^\beta - 1\right|^{p-1}}{\left|\frac{x}{|x|} - z\right|^{N+sp}}.$$

Making the change of variables $z = \frac{y}{|x|}$, we have that

$$F(x) = \int\limits_{B_\varepsilon} k\left(x, \tfrac{y}{|x|}\right)dy\,|x|^{\beta(p-1)-N-sp} = \int\limits_{B_{\frac{\varepsilon}{|x|}}} k(x,z)dz\,|x|^{\beta(p-1)-sp}. \tag{31}$$

On the other hand, since $|x|>r$ and $\beta<0$, we have that $\left(\frac{\varepsilon}{|x|}\right)^{\beta} \geq \left(\frac{\varepsilon}{r}\right)^{\beta}$, and therefore

$$k(x,z) \leq \frac{\left||z|^{\beta}-1\right|^{p-1} - \left|\left(\frac{\varepsilon}{r}\right)^{\beta}-1\right|^{p-1}}{\left|\frac{x}{|x|}-z\right|^{N+sp}}.$$

Thus, by a simple change of variable and by rotation invariance, we have that

$$0 \leq \int\limits_{B_{\frac{\varepsilon}{|x|}}} k(x,z)dz \leq \int\limits_{B_{\frac{\varepsilon}{|x|}}} \frac{\left||z|^{\beta}-1\right|^{p-1} - \left|\left(\frac{\varepsilon}{r}\right)^{\beta}-1\right|^{p-1}}{|e_1-z|^{N+sp}} dz. \tag{32}$$

Now, since $|e_1-z| \geq 1-\frac{\varepsilon}{r}$ for any $z \in B_{\frac{\varepsilon}{|x|}}$, $p>1$, $\beta \in \left(-\frac{N}{p-1}, \frac{ps-N}{p-1}\right)$, we get

$$\begin{aligned} 0 &\leq \int\limits_{B_{\frac{\varepsilon}{|x|}}} \frac{\left||z|^{\beta}-1\right|^{p-1} - \left|\left(\frac{\varepsilon}{r}\right)^{\beta}-1\right|^{p-1}}{|e_1-z|^{N+sp}} dz \\ &\leq \frac{1}{\left(1-\frac{\varepsilon}{r}\right)^{N+sp}} \int\limits_{B_{\frac{\varepsilon}{|x|}}} |z|^{\beta(p-1)}dz = \frac{\alpha_N}{\left(1-\frac{\varepsilon}{r}\right)^{N+sp}} \left(\frac{\varepsilon}{r}\right)^{\beta(p-1)+N}. \end{aligned} \tag{33}$$

Then, by (31), (32), and (33), we have that

$$0 \leq F(x) \leq \frac{\alpha_N}{\left(1-\frac{\varepsilon}{r}\right)^{N+sp}} \left(\frac{\varepsilon}{r}\right)^{\beta(p-1)+N} |x|^{\beta(p-1)-sp}, \tag{34}$$

for any $x \in \mathbb{R}^N \setminus B_r$.

By (29), (30), and (34), we obtain

$$\begin{aligned} &\int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(\phi_\varepsilon(x)-\phi_\varepsilon(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} dxdy \leq \\ &\leq \left(\mathcal{C}(\beta) + \frac{2\alpha_N}{\left(1-\frac{\varepsilon}{r}\right)^{N+sp}} \left(\frac{\varepsilon}{r}\right)^{\beta(p-1)+N}\right) \int\limits_{\mathbb{R}^N\setminus B_r} |x|^{\beta(p-1)-sp}\varphi(x)dx. \end{aligned}$$

Thus, by (2), we get

$$\int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(\phi_\varepsilon(x)-\phi_\varepsilon(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} dxdy \leq 0$$

for any ε small enough. □

Remark 4.1. *It is worth noting that although Lemma 4.1 establishes that $\varphi_\varepsilon(x) = \max\{|x|^\beta, \varepsilon^\beta\}$ is $(-\Delta_p)^s$ non-positive in the* weak sense, *a similar conclusion holds in the viscosity sense by standard arguments. Specifically, since both $|x|^\beta$ and ε^β are $(-\Delta_p)^s$ non-positive in the viscosity sense, φ_ϵ inherits this property. Any viscosity test function of φ_ϵ can be viewed as a test function for either $|x|^\beta$ or ϵ^β, ensuring the desired inequality. This connection helps bridge the weak and viscosity frameworks and aligns with intuition from viscosity theory.*

Before showing our next result, we define

$$\mathcal{A}_r := A_{\frac{r}{2},2r}, \quad r_\varepsilon := r\left(\frac{\varepsilon}{1+\varepsilon 2^\beta}\right)^{-\frac{1}{\beta}},$$

and

$$\psi_\varepsilon(x) \coloneqq \begin{cases} r_\varepsilon^\beta & \text{if } 0 \le |x| \le r_\varepsilon, \\ |x|^\beta & \text{if } r_\varepsilon < |x| \le 2r, \\ (2r)^\beta & \text{if } 2r < |x|. \end{cases}$$

Observe that $r_\varepsilon < \frac{r}{2}$ and $r_\varepsilon \to 0^+$ as $\varepsilon \to 0^+$.

Lemma 4.2. *There is $\varepsilon_0 \in (0,1)$ independent of r such that for any $\varepsilon \in (0, \varepsilon_0)$, ψ_ε is a weak solution of*

$$(-\Delta_p)^s \psi_\varepsilon(x) \le 0 \ \text{ in } \mathcal{A}_r.$$

Proof. Let $\varphi \in C_c^\infty(\mathcal{A}_r)$ be non-negative. Then

$$\int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(\psi_\varepsilon(x) - \psi_\varepsilon(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy = J_1 + 2J_2, \tag{35}$$

where

$$J_1 = \int\limits_{\mathcal{A}_r^2} \frac{\Psi_p(\psi_\varepsilon(x) - \psi_\varepsilon(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy,$$

$$J_2 = \int\limits_{\mathcal{A}_r} \int\limits_{\mathbb{R}^N \setminus \mathcal{A}_r} \frac{\Psi_p(\psi_\varepsilon(x) - \psi_\varepsilon(y))}{|x-y|^{N+sp}} \varphi(x) dydx.$$

Observe that

$$\mathbb{R}^{\mathbb{N}} \setminus \mathcal{A}_r = C_r^1 \cup C_r^2 \cup C_r^3$$

with

$$C_r^1 \coloneqq A_{r_\varepsilon, \frac{r}{2}}, \quad C_r^2 \coloneqq \mathbb{R}^N \setminus B_{2r} \quad \text{and} \quad C_r^3 \coloneqq B_{r_\varepsilon}.$$

Then

$$\begin{aligned} J_2 = & \int\limits_{\mathcal{A}_r} \int\limits_{C_r^1} \frac{\Psi_p(|x|^\beta - |y|^\beta)}{|x-y|^{N+sp}} dy \, \varphi(x) dx \\ & + \int\limits_{\mathcal{A}_r} \int\limits_{C_r^2} \frac{||x|^\beta - (2r)^\beta|^{p-1}}{|x-y|^{N+sp}} dy \, \varphi(x) dx - \int\limits_{\mathcal{A}_r} \int\limits_{C_r^3} \frac{|r_\varepsilon^\beta - |x|^\beta|^{p-1}}{|x-y|^{N+sp}} dy \, \varphi(x) dx. \end{aligned}$$

Thus

$$J_1 + 2J_2 = \int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(|x|^\beta - |y|^\beta)(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy + 2 \int\limits_{\mathcal{A}_r} (G_1(x) + G_2(x)) \, \varphi(x) dx,$$

where

$$G_1(x) = \int\limits_{C_r^2} \frac{||x|^\beta - (2r)^\beta|^{p-1} - ||x|^\beta - |y|^\beta|^{p-1}}{|x-y|^{N+sp}} dy,$$

$$G_2(x) = \int\limits_{C_r^3} \frac{||y|^\beta - |x|^\beta|^{p-1} - |r_\varepsilon^\beta - |x|^\beta|^{p-1}}{|x-y|^{N+sp}} dy.$$

By Theorem 3.1, we have

$$J_1 + J_2 \le 2 \int\limits_{\mathcal{A}_r} (G_1(x) + G_2(x)) \, \varphi(x) dx. \tag{36}$$

On the other hand, for any $x \in \mathcal{A}_r$

$$G_2(x) \le \alpha_N \frac{r_\varepsilon^{\beta(p-1)+N}}{r^{N+sp}}, \tag{37}$$

by calculation as in the proof of Lemma 4.1.

Now, taking $x \in \mathcal{A}_r$, we get

$$G_1(x) = \int\limits_{C_r^2} \frac{||x|^\beta - (2r)^\beta|^{p-1} - ||x|^\beta - |y|^\beta|^{p-1}}{|x-y|^{N+sp}} dy$$
$$= \int\limits_{C_r^2} \frac{\left|1 - \left(\frac{2r}{|x|}\right)^\beta\right|^{p-1} - \left|1 - \left(\frac{|y|}{|x|}\right)^\beta\right|^{p-1}}{\left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{N+sp}} dy |x|^{\beta(p-1)-N-sp}.$$

Making the change of variable $z = \frac{y}{|x|}$, we obtain that

$$G_1(x) = \int\limits_{\mathbb{R}^N \setminus B_{\frac{2r}{|x|}}} \frac{\left|1 - \left(\frac{2r}{|x|}\right)^\beta\right|^{p-1} - \left|1 - |z|^\beta\right|^{p-1}}{\left|\frac{x}{|x|} - z\right|^{N+sp}} dy |x|^{\beta(p-1)-sp}.$$

Since $x \in \mathcal{A}_r$ and $\beta < 0$, we have that $2 < \frac{2r}{|x|} < 4$ and

$$G_1(x) \le \int\limits_{\mathbb{R}^N \setminus B_4} \frac{\left|1 - \left(\frac{2r}{|x|}\right)^\beta\right|^{p-1} - \left|1 - |z|^\beta\right|^{p-1}}{\left|\frac{x}{|x|} - z\right|^{N+sp}} dy \left(\frac{r}{2}\right)^{\beta(p-1)-sp}$$
$$\le \int\limits_{\mathbb{R}^N \setminus B_4} \frac{\left|1 - 4^\beta\right|^{p-1} - \left|1 - |z|^\beta\right|^{p-1}}{\left|\frac{x}{|x|} - z\right|^{N+sp}} dy \left(\frac{r}{2}\right)^{\beta(p-1)-sp}$$

As the last integration is invariant under the rotation of coordinate axes, we get

$$G_1(x) \le D_N r^{\beta(p-1)-sp} \quad \forall x \in \mathcal{A}_r, \tag{38}$$

here D_N denotes a negative constant depending only on N.

Finally by (35), (36), (38) and (37),

$$\int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(\psi_\varepsilon(x) - \psi_\varepsilon(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy$$
$$\le \left(D_N r^{\beta(p-1)-sp} + \alpha_N \frac{r_\varepsilon^{\beta(p-1)+N}}{r^{N+sp}} \right) \int\limits_{\mathcal{A}_r} \varphi(x)\, dx$$
$$\le \left(D_N + \alpha_N \left(\frac{\varepsilon}{1+\varepsilon 2^\beta} \right)^{-(p-1)-\frac{N}{\beta}} \right) r^{\beta(p-1)-sp} \int\limits_{\mathcal{A}_r} \varphi(x)\, dx.$$

Since $D_N < 0$ and $\left(\frac{\varepsilon}{1+\varepsilon 2^\beta}\right)^{-(p-1)-\frac{N}{\beta}} \to 0^+$ as $\varepsilon \to 0$, there is a positive ε_0 independent of r such that

$$\int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(\psi_\varepsilon(x) - \psi_\varepsilon(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy \le 0.$$

□

4.2. **Hadamard-type results.** We are now in a position to prove our Hadamard properties.

Lemma 4.3. *Let $N \geq 2$, $0 < s < 1$ and $1 < p < \infty$. If $N > ps$ then for any $\beta \in \left(-\frac{N}{p-1}, \frac{ps-N}{p-1}\right)$ and any $r_0 > 1$ there is a positive constant $C > 0$ such that for any $u \not\equiv 0$ non-negative lower semi-continuous weak solution of*

$$(-\Delta_p)^s u \geq 0 \quad \text{in } \mathbb{R}^N,$$

we have that

$$m(r) \geq C m(r_0) r^\beta \quad \forall r > r_0,$$

where $m(r) := \min\{u(x) \colon x \in \overline{B_r}(0)\}$.

Proof. Let $r_0 > 1$. By Lemma 4.1, there exist $\varepsilon_0 > 0$ such that ϕ_ε is a weak solution of

$$(-\Delta_p)^s \phi_\varepsilon(x) \leq 0 \quad \text{in } A_{r_0,R} \tag{39}$$

for any $R > r_0$. For any $\varepsilon > \varepsilon_0$ and $R > r_0$, we define

$$H_{\varepsilon,R}(x) := \frac{m(r_0)}{\varepsilon^\beta - R^\beta} \begin{cases} \phi_\varepsilon(x) - R^\beta & \text{if } |x| \geq \varepsilon, \\ 0 & \text{if } |x| \leq \varepsilon. \end{cases}$$

Observe that $H_{\varepsilon,R}$ is a weak solution of (39) and $u \geq H_{\varepsilon,R}$ in $\mathbb{R}^N \setminus A_{r_0,R}$. Thus, by the comparison principle (see, for instance, [14]), we have that $u \geq H_{\varepsilon,R}$ in $\mathbb{R}^N$. Then

$$u(x) \geq \frac{m(r_0)}{\varepsilon^\beta - R^\beta}(|x|^\beta - R^\beta) \text{ in } A_{r_0,R}.$$

Passing to limits as $R \to \infty$, we have that

$$u(x) \geq \frac{m(r_0)}{\varepsilon^\beta}|x|^\beta \text{ in } \{x \in \mathbb{R}^N \colon |x| > r_0\}.$$

Finally, taking $C = \varepsilon_0^{-\beta}$ we have

$$m(r) \geq C m(r_0) r^\beta \quad \forall r \geq r_0.$$

□

Our second Hadamard property is

Lemma 4.4. *Let $N \geq 2$, $0 < s < 1$ and $1 < p < \infty$. If $N > ps$ then there is a positive constant $C > 0$ such that for any $u \not\equiv 0$ non-negative lower semi-continuous weak solution of*

$$(-\Delta_p)^s u \geq 0 \quad \text{in } \mathbb{R}^N$$

we have that

$$m(\tfrac{r}{2}) \leq C m(r) \quad \forall r > 1,$$

where $m(r) := \min\{u(x) \colon x \in \overline{B_r}(0)\}$.

Proof. Let $r > 1$ and $\beta \in \left(-\frac{N}{p-1}, \frac{ps-N}{p-1}\right)$. By Lemma 4.2, there is $\varepsilon_0 > 0$ independent of r such that ψ_ε is a weak solution of

$$(-\Delta_p)^s \psi_\varepsilon(x) \leq 0 \quad \text{in } \mathcal{A}_r. \tag{40}$$

For any $\varepsilon > \varepsilon_0$, we define

$$J_{\varepsilon,r}(x) := m\left(\tfrac{r}{2}\right) \frac{\psi_\varepsilon(x) - (2r)^\beta}{r_\varepsilon^\beta - (2r)^\beta}.$$

Observe that $J_{\varepsilon,r}$ is also weak solution of (40) and $u \geq J_{\varepsilon,r}$ in $\mathbb{R}^N \setminus \mathcal{A}_r$. Thus, by the comparison principle, we have that $u \geq J_{\varepsilon,r}$ in $\mathbb{R}^N$. Then

$$m(r) \geq m\left(\tfrac{r}{2}\right) \min\left\{ \frac{\psi_\varepsilon(x) - (2r)^\beta}{r_\varepsilon^\beta - (2r)^\beta} : |x| = r \right\} = m\left(\tfrac{r}{2}\right) \frac{r^\beta - 2^\beta r^\beta}{r^\beta \left(\frac{\varepsilon}{1+\varepsilon 2^\beta}\right)^{-1} - 2^\beta r^\beta}$$

$$= m\left(\tfrac{r}{2}\right) \varepsilon(1 - 2^\beta).$$

Finally, taking $C = \varepsilon(1 - 2^\beta)$ we have

$$m(r) \geq C m\left(\tfrac{r}{2}\right).$$

□

5. A Liouville-type theorem

We now prove a Liouville-type theorem. We split the proof into two cases.

5.1. **Case** $N < sp$**.** Let $\beta \in \left(0, \frac{ps-N}{p-1}\right)$, $0 < \varepsilon < 1 < r < R$, and u be a non-negative lower semi-continuous weak solution of

$$(-\Delta_p)^s u \geq 0 \quad \text{in } \mathbb{R}^N. \tag{41}$$

We know, by Theorem 1.1, $v_\beta(x) = |x|^\beta$ is a weak solution of

$$(-\Delta_p)^s v_\beta(x) = \mathcal{C}(\beta)|x|^{\beta(p-1)-sp} \quad \text{in } \mathbb{R}^N \setminus \{0\} \tag{42}$$

with $\mathcal{C}(\beta) > 0$ (see (2)). We now define

$$\theta_\beta^\varepsilon(x) \coloneqq m(r) \begin{cases} 1 & \text{if } 0 \leq |x| < \varepsilon, \\ \frac{R^\beta - |x|^\beta}{R^\beta - \varepsilon^\beta} & \text{if } \varepsilon \leq |x| < R, \\ 0 & \text{if } R \leq |x|, \end{cases}$$

where $m(r) \coloneqq \min\{u(x) : x \in \overline{B_r}(0)\}$.

First, we prove the following auxiliary result.

Lemma 5.1. *For ε sufficiently small, θ_β^ε is a weak solution of*

$$(-\Delta_p)^s \theta_\beta^\varepsilon(x) \leq 0 \quad \text{in } A_{r,R}. \tag{43}$$

Proof. Let $\varphi \in C_0^\infty(A_{r,R})$ be non-negative. Then

$$\int_{\mathbb{R}^{2N}} \frac{\Psi_p(\theta_\beta^\varepsilon(x) - \theta_\beta^\varepsilon(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy = I_1 + I_2 + I_3,$$

where

$$I_1 = -\frac{2m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int_{A_{r,R}} \int_{B_\varepsilon(0)} \frac{||x|^\beta - \varepsilon^\beta|^{p-1}}{|x-y|^{N+sp}} \varphi(x) dydx$$

$$I_2 = \frac{2m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int_{A_{r,R}} \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|R^\beta - |x|^\beta|^{p-1}}{|x-y|^{N+sp}} \varphi(x) dydx$$

$$I_3 = \frac{m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int_{A_{\varepsilon,R}^2} \frac{\Psi_p(|y|^\beta - |x|^\beta)(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy.$$

Observe that

$$I_3 = \frac{m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int_{\mathbb{R}^{2N}} \frac{\Psi_p(|y|^\beta - |x|^\beta)(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy + J_1 + J_2$$

with

$$J_1 = \frac{2m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int\limits_{A_{r,R}} \int\limits_{B_\varepsilon(0)} \frac{||x|^\beta - |y|^\beta|^{p-1}}{|x-y|^{N+sp}} \varphi(x) dy dx,$$

$$J_2 = -\frac{2m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int\limits_{A_{r,R}} \int\limits_{\mathbb{R}^N \setminus B_R(0)} \frac{||y|^\beta - |x|^\beta|^{p-1}}{|x-y|^{N+sp}} \varphi(x) dy dx.$$

Then, using that $v_\beta(x) = |x|^\beta$ is a weak solution of (42), we have that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(\theta_\beta^\varepsilon(x) - \theta_\beta^\varepsilon(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} dx dy = \\
&= \frac{2m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int\limits_{A_{r,R}} \int\limits_{B_\varepsilon(0)} \frac{||x|^\beta - |y|^\beta|^{p-1} - ||x|^\beta - \varepsilon^\beta|^{p-1}}{|x-y|^{N+sp}} \varphi(x) dy dx \\
&- \frac{2m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int\limits_{A_{r,R}} \int\limits_{\mathbb{R}^N \setminus B_R(0)} \frac{||y|^\beta - |x|^\beta|^{p-1} - |R^\beta - |x|^\beta|^{p-1}}{|x-y|^{N+sp}} \varphi(x) dy dx \\
&- \mathcal{C}(\beta) \frac{m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int\limits_{A_{r,R}} |x|^{\beta(p-1)-sp} \varphi(x) dx \\
&\le \frac{2m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int\limits_{A_{\varepsilon,R}} \int\limits_{B_\varepsilon(0)} \frac{||x|^\beta - |y|^\beta|^{p-1} - ||x|^\beta - \varepsilon^\beta|^{p-1}}{|x-y|^{N+sp}} \varphi(x) dy dx \\
&- \mathcal{C}(\beta) \frac{m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \int\limits_{A_{\varepsilon,R}} |x|^{\beta(p-1)-sp} \varphi(x) dx.
\end{aligned}$$

Note that, for any $x \in A_{r,R}$, using the change of variable $z = \frac{y}{|x|}$, we have

$$\begin{aligned}
&\int\limits_{B_\varepsilon(0)} \frac{||x|^\beta - |y|^\beta|^{p-1} - ||x|^\beta - \varepsilon^\beta|^{p-1}}{|x-y|^{N+sp}} dy = \\
&\quad = \int\limits_{B_\varepsilon(0)} \frac{|1 - \left(\frac{|y|}{|x|}\right)^\beta|^{p-1} - |1 - \left(\frac{\varepsilon}{|x|}\right)^\beta|^{p-1}}{\left|\left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{N+sp}} dy |x|^{\beta(p-1)-N-sp} \\
&\quad = \int\limits_{B_{\frac{\varepsilon}{|x|}}(0)} \frac{|1 - |z|^\beta|^{p-1} - |1 - \left(\frac{\varepsilon}{|x|}\right)^\beta|^{p-1}}{\left|\frac{x}{|x|} - z\right|^{N+sp}} dz |x|^{\beta(p-1)-sp} \\
&\quad \le \frac{\alpha_N \varepsilon^N}{\left(1 - \frac{\varepsilon}{R}\right)^{N+sp}} |x|^{\beta(p-1)-sp}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{2N}} \frac{\Psi_p(\theta_\beta^\varepsilon(x) - \theta_\beta^\varepsilon(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} dx dy = \\
&\quad \le \frac{m(r)^{p-1}}{|R^\beta - \varepsilon^\beta|^{p-1}} \left(\frac{2\alpha_N \varepsilon^N}{(1 - \frac{\varepsilon}{R})^{N+sp}} - \mathcal{C}(\beta) \right) \int\limits_{A_{\varepsilon,R}} |x|^{\beta(p-1)-sp} \varphi(x) dx, < 0
\end{aligned}$$

provided ε is small enough. □

We now prove our Liouville-type theorem for the case $sp > N$.

Theorem 5.1. *Let $N \geq 2, 0 < s < 1$, and $1 < p < \infty$. If $N < ps$ and u is a non-negative lower semi-continuous weak solution of (41), then u is constant.*

Proof. Let $\beta \in \left(0, \frac{ps-N}{p-1}\right)$ and $0 < \varepsilon < 1 < r < R$. By Lemma 5.1, for ε sufficiently small, $\theta^{\varepsilon}_{\beta}$ is a weak solution of (43) and it is easy to see that $\theta^{\varepsilon}_{\beta} \leq u$ in $\mathbb{R}^N \setminus A_{r,R}$. Thus, by the comparison principle, we have that $\theta^{\varepsilon}_{\beta} \leq u$ in $\mathbb{R}^N$. Therefore $m(r) \leq u(x)$ for any $|x| \geq r$. Then there is $x_0 \in \overline{B_r}(0)$ such that $u(x_0) \leq u(x)$ for any $x \in \mathbb{R}^N$.

On the other hand, by [26] (see also [3]), we know the u is a viscosity solution of (41). Finally, since u attains its minimum, we can conclude that u is constant. □

5.2. **Cases $N = sp$.** Let $0 < \varepsilon < 1 < r < R$. In this case, we take a non-negative function $\zeta_\varepsilon \in C^\infty_c(\Omega)$ such that

$$\zeta_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in B_{\frac{\varepsilon}{2}}(0), \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_\varepsilon(0), \end{cases}$$

and

$$\xi_\varepsilon(x) := \begin{cases} \log(|x|) - \log(\varepsilon) + \kappa\zeta_\varepsilon(x) & \text{if } x \in B_\varepsilon(0), \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_\varepsilon(0), \end{cases}$$

where κ is a positive constant to be chosen later.

By Theorem 1.1 and [24, Lemma 2.8], we have that $\rho_\varepsilon(x) := \xi_\varepsilon(x) - \log(|x|)$ satisfies

$$(-\Delta_p)^s \rho_\varepsilon(x) = h(x) \quad \text{in } A_{r,R}$$

where for a.e. Lebesgue point $x \in A_{r,R}$

$$h(x) := 2\int_{B_\varepsilon(0)} \frac{F(x,y)}{|x-y|^{N+sp}}dy$$

with

$$F(x,y) := \Psi_p\left(-\log\left(|x|\right) + \log\left(|y|\right) - \xi_\varepsilon\left(y\right)\right) - \Psi_p\left(-\log\left(|x|\right) + \log\left(|y|\right)\right).$$

Observe that, for a.e. Lebesgue point $x \in A_{r,R}$ and for any $y \in B_\varepsilon(0)$, we have

$$\begin{aligned} F(x,y) &= \Psi_p(-\log(|x|) + \log(\varepsilon) - \kappa\zeta_\varepsilon(y)) + (\log(|x|) - \log(|y|)|^{p-1} \\ &= (\log(|x|) - \log(|y|)|^{p-1} - (\log(|x|) - \log(\varepsilon) + \kappa\xi_\varepsilon(y))^{p-1}. \end{aligned}$$

Then for a.e. Lebesgue point $x \in A_{r,R}$

$$h(x) = 2\int_{B_\varepsilon(0)} \frac{(\log(|x|) - \log(|y|))^{p-1} - (\log(|x|) - \log(\varepsilon) + \kappa\xi_\varepsilon(y))^{p-1}}{|x-y|^{N+sp}}dy.$$

Now we choose κ large enough so that $h(x) \leq 0$ for a.e. Lebesgue point $x \in A_{r,R}$. Then, we can prove the second case of our Liouville-type theorem.

Theorem 5.2. *Let $N \geq 2, 0 < s < 1$, and $1 < p < \infty$. If $N = ps$ and u is a non-negative lower semi-continuous weak solution of (41), then u is constant.*

Proof. Let u be a non-negative lower semi-continuous weak solution of (41) and

$$\theta_\varepsilon(x) = m(r)\frac{\rho_\varepsilon(x) + \log(R)}{-\log(\varepsilon) + \kappa + \log(R)}$$

where $m(r) := \min\{u(x) \colon x \in \overline{B_r}(0)\}$. Then θ_ε is a weak solution of (43) and it is easy to see that $\theta_\varepsilon \leq u$ in $\mathbb{R}^N \setminus A_{r,R}$. Thus, by the comparison principle, we have that $\theta_\varepsilon \leq u$ in $\mathbb{R}^N$. Therefore $m(r) \leq u(x)$ for any $|x| \geq r$. Then there is $x_0 \in \overline{B_r}(0)$ such that $u(x_0) \leq u(x)$ for any $x \in \mathbb{R}^N$.

On the other hand, by [26] (see also [3]), we know that u is a viscosity solution of (41). Finally, since u attains its minimum, we can conclude that u is constant. □

6. A nonlinear Liouville-type theorem

In this last section, we prove our non-linear Liouville-type theorem (see Theorem 1.3). As before, we split the proof in two cases.

6.1. The sub-critical case.

First, we show the following result.

Theorem 6.1. *Let $N \geq 2, 0 < s < 1$, $1 < p < \infty$, and $N > ps$. If $0 < q < \frac{N(p-1)}{N-ps}$ and $u \in C(\mathbb{R}^N)$ is a non-negative weak solution of*

$$(-\Delta_p)^s u - u^q \geq 0 \quad \text{in } \mathbb{R}^N. \tag{44}$$

then $u \equiv 0$.

Proof. Let u be a non-negative lower semi-continuous weak solution of (44). By [3], u is a viscosity solution of (44).

We suppose by contradiction that $u \not\equiv 0$ in $\mathbb{R}^N$. By [14, Theorem 1.2], we have that $u > 0$ a.e. in $\mathbb{R}^N$.

On the other hand, by [26] (see also [3]), we know the u is a viscosity solution of

$$(-\Delta_p)^s u(x) \geq 0 \quad \text{in } \mathbb{R}^N.$$

Therefore $u > 0$ in $\mathbb{R}^N$.

Let's take a function $\mu \in C^\infty([0,\infty), \mathbb{R})$ such that μ is non-increasing, $0 \leq \mu \leq 1$, and

$$\mu(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if } t \geq 1. \end{cases}$$

Then, by [24, Proposition 2.12], there is a positive constant C such that $w(x) = \mu(|x|)$ satisfies in strongly sense

$$(-\Delta_p)^s w(x) \leq C \quad \text{in } B_1(0).$$

Now, for any $R > 1$, we take

$$\zeta(x) = m\left(\tfrac{R}{2}\right) \mu\left(\tfrac{|x|}{R}\right),$$

where $m\left(\frac{R}{2}\right) = \min\left\{u(x) \colon x \in \overline{B_{\frac{R}{2}}}(0)\right\}$. Observe that ζ satisfies in strongly sense

$$(-\Delta_p)^s \zeta(x) \leq m\left(\tfrac{R}{2}\right)^{p-1} \frac{C}{R^{ps}} \quad \text{in } B_R(0). \tag{45}$$

Here, C is a positive constant independent of R.

On the other hand, since $\zeta(x) \leq u(x)$ in $\mathbb{R}^N \setminus A_{\frac{R}{2},R}$ and u is lower semi-continuous function, there is $x_R \in B_R(0)$ such that $u(x_R) - \zeta(x_R) \leq u(x) - \zeta(x)$ for any $x \in \mathbb{R}^N$.

We divide the rest of the proof into two cases.

Case 1: $\frac{R}{2} < |x_R| < R$. We take $r \ll \operatorname{dist}(x_R, \partial B_R(0))$, and

$$\phi_r(x) = \begin{cases} \zeta(x) - \zeta(x_R) + u(x_R) & \text{if } x \in B_r(x_R), \\ u(x) & \text{if } x \in \mathbb{R}^N \setminus B_r(x_R). \end{cases}$$

Note that $\phi_r(x) \in C^\infty(B_r(x_R))$, $\phi_r(x_R) = u(x_R)$, $\phi_r(x) \leq u(x)$ in $B_r(x_R)$ and $\nabla\phi_r(x_R) = \nabla\zeta_r(x_R) \neq 0$. Then, since u is a viscosity solution of (44), we have that

$$u(x_R)^q \leq (-\Delta_p)^s \phi_r(x_R).$$

Now, using that $u(x_R)-\zeta(x_R)\le u(x)-\zeta(x)$ for any $x\in\mathbb{R}^N$, $x_R\in B_R(0)$, and (45), we get

$$m(R)^q\le(-\Delta_p)^s\phi_r(x_R)\le(-\Delta_p)^s\zeta_r(x_R)\le m\left(\tfrac{R}{2}\right)^{p-1}\frac{C}{R^{ps}}.$$

Then, by Lemma 4.4, we have

$$m(R)^q\le\frac{C}{R^{ps}}m(R)^{p-1}\quad\forall R>1$$

where C is a positive constant independent of R. If $0<q\le p-1$, we obtain a contradiction. On the other hand, if $q>p-1$, we have

$$m(R)\le CR^{\kappa}$$

where $\kappa=-\frac{ps}{q-p+1}$. Since $p-1<q<\frac{N(p-1)}{N-ps}$, there is $\beta\in(\frac{N}{p-1},\frac{ps-N}{p-1})$ such that $\kappa<\beta$. Then, by Lemma 39, there is $r_0>1$ and a positive constant such that

$$m(R)\le CR^{\kappa}\le CR^{\kappa-\beta}m(R)\quad\forall R>r_0.$$

We again obtain a contradiction.

Case 2: $|x_R|\le\frac{R}{2}$. Then

$$0\le u(x_R)-m(\tfrac{R}{2})=u(x_R)-\zeta(x_R)\le u(x)-\zeta(x)\quad\forall x\in\mathbb{R}^N.$$

In particular, if we $\tilde{x}\in B_{\frac{R}{2}}(0)$ so that $u(\tilde{x})=m(\frac{R}{2})$ we have that

$$0\le u(x_R)-m(\tfrac{R}{2})\le 0.$$

Therefore $u(x_R)=m(\frac{R}{2})$ and $\zeta(x)\le u(x)$ for any $x\in\mathbb{R}^N$.

Thus if $p>\frac{2}{2-s}$, we can proceed as in the previous case. But if $1<p\le\frac{2}{2-s}$, we have a problem because x_R is a critical point of ζ but it is not isolated. Then ϕ_r is not an admissible test function. To solve this problem, we take

$$\tilde{\phi}_r(x)=\begin{cases}\zeta(x)-m\left(\frac{R}{2}\right)|x-x_R|^{\gamma} & \text{if } x\in B_r(x_R),\\ u(x) & \text{if } x\in B_r(x_R),\end{cases}$$

as a test function with

$$\gamma>\tfrac{sp}{p-1}\quad\text{ and }\quad r<R^{-\frac{sp}{\gamma(p-1)-sp}}.$$

Observe that

$$\begin{aligned}m(R)^q\le&u(x_R)^q\le(-\Delta_p)^s\tilde{\phi}_r(x_R)\\ &\le\int_{B_r(x_R)}\frac{|\zeta(x_R)-\zeta(x)+m\left(\frac{R}{2}\right)|x-x_R|^{\gamma}|^{p-1}}{|x-x_R|^{N+sp}}dx\\ &\quad+\int_{\mathbb{R}^N\setminus B_r(x_R)}\frac{|u(x_R)-u(x)|^{p-2}(u(x_R)-u(x))}{|x-x_R|^{N+sp}}dx.\end{aligned}$$

Note that, $0<p-1\le\frac{s}{2-s}<1$ and $\zeta(x_R)-\zeta(x)\ge0$ for any $x\in\mathbb{R}^N$. Then, using that

$$(a+b)^q\le a^q+b^q\quad\forall\, a,b\ge0\quad q\in(0,1]$$

(see [24]), $\zeta(x) \le u(x)$ for any $x \in \mathbb{R}^N$, and (45), we have that

$$\begin{aligned} m(R)^q \le & u(x_R)^q \le (-\Delta_p)^s \tilde{\phi}_r(x_R) \\ \le & m\left(\tfrac{R}{2}\right)^{p-1} \int_{B_r(x_R)} \frac{|x-x_R|^{\gamma(p-1)}}{|x-x_R|^{N+sp}} dx + (-\Delta_p)^s \zeta(x_R) \\ \le & C\left(r^{\gamma(p-1)-sp} + \tfrac{1}{R^{ps}}\right) m\left(\tfrac{R}{2}\right)^{p-1} \\ \le & \frac{C}{R^{ps}} m\left(\tfrac{R}{2}\right)^{p-1} \qquad \forall R > 1, \end{aligned}$$

where C is a positive constant independent of R.

Now the proof follows exactly the proof of Case 1. □

6.2. **The super-critical.** To conclude this article, we prove the following result.

Theorem 6.2. *Let $N \ge 2, 0 < s < 1$, $1 < p < \infty$, and $N > ps$. If $q > \frac{N(p-1)}{N-ps}$ then there is a positive solution of (44).*

Proof. In this case, we take $\kappa = \frac{sp}{q-p+1}$ and

$$w(x) = \frac{1}{(1+|x|)^{\kappa}}.$$

Observe that, since $q > \frac{N(p-1)}{N-ps}$ we have that $0 < \kappa < \frac{N-ps}{p-1}$.

For any $x \in \mathbb{R}^N \setminus \{0\}$, we have that

$$\begin{aligned} 2\int_{\mathbb{R}^N} \frac{\Psi_p(w(x)-w(y))}{|x-y|^{N+sp}} dy &= 2\int_{\mathbb{R}^N} \frac{\Psi_p\left(\frac{1}{(1+|x|)^{\kappa}} - \frac{1}{(1+|y|)^{\kappa}}\right)}{|x-y|^{N+sp}} dy \\ &= 2\frac{1}{(1+|x|^2)^{\kappa(p-1)+sp+N}} \int_{\mathbb{R}^N} \frac{\Psi_p\left(1-\left(\frac{1+|x|}{1+|y|}\right)^{\kappa}\right)}{\left|\frac{x}{1+|x|} - \frac{y}{1+|x|}\right|^{N+sp}} dy. \end{aligned}$$

Applying a rotation, we may assume that $\frac{x}{|x|} = e_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$, and via the change of variable $z = \frac{1}{1+|x|}\left(y + \frac{x}{|x|}\right)$, and since

$$\frac{x}{1+|x|} - \frac{y}{1+|x|} = \frac{x}{|x|} - \frac{x}{(1+|x|)|x|} - \frac{y}{1+|x|} = \frac{x}{|x|} - \frac{1}{1+|x|}\left(y + \frac{x}{|x|}\right),$$

and $\kappa(p-1) + sp = \kappa q$, we get

$$2(1+|x|)^{\kappa q} \int_{\mathbb{R}^N} \frac{\Psi_p(w(x)-w(y))}{|x-y|^{N+sp}} dy = 2\int_{\mathbb{R}^N} \frac{\Psi_p\left(1-\left(\frac{1+|x|}{1+|(1+|x|)z-e_1|}\right)^{\kappa}\right)}{|e_1-z|^{N+sp}} dz.$$

Now, using that

$$(1+|x|)|z| \le 1 + |(1+|x|)z - e_1|$$

we have

$$\int_{\mathbb{R}^N} \frac{\Psi_p\left(1-\left(\frac{1+|x|}{1+|(1+|x|)z-e_1|}\right)^{\kappa}\right)}{|e_1-z|^{N+sp}} dz \ge \int_{\mathbb{R}^N} \frac{\Psi_p\left(1-\frac{1}{|z|^{\kappa}}\right)}{|e_1-z|^{N+sp}} dz = \mathcal{C}(-\kappa).$$

Since, $0 > -\kappa > \frac{ps-N}{p-1}$, by (2), we have that $\mathcal{C}(-\kappa) > 0$. See also Remark 1.3. Then,

$$\mathcal{C}(-\kappa)^{\frac{1}{q-p+1}} w,$$

is a positive solution of (44). □

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