# MIXED LOCAL AND NONLOCAL LAPLACIAN WITHOUT STANDARD CRITICAL EXPONENT FOR LANE-EMDEN EQUATION

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Abstract. In this paper, we investigate a mixed elliptic equation involving both local and nonlocal Laplacian operators, with a power-type nonlinearity. Specifically, we consider a Lane-Emden type equation of the form

$$-\Delta u + (-\Delta)^s u = u^p, \qquad \text{in } \mathbb{R}^n.$$

where the operator combines the classical Laplacian and the fractional Laplacian. We establish the existence of solutions for exponents slightly below the critical local Sobolev exponent, that is, for $p < \frac{n+2}{n-2}$, with $p$ close to $\frac{n+2}{n-2}$.

Our results show that, due to the interaction between the local and nonlocal operators, this mixed Lane-Emden–Fowler equation does not admit a critical exponent in the traditional sense. The existence proof is carried out using a Lyapunov–Schmidt type reduction method and, as far as we know, provide the first example of an elliptic operator for which the duality between critical exponents fails.

## 1. Introduction

The objective of this paper is to explore critical exponents associated with a Lane-Emden type equation involving a mixed elliptic operator of the form

$$L_{a,b}(u) \coloneqq -a\Delta u + b(-\Delta)^s u = u^p,$$

with $a, b \in [0, \infty)$, with at least one of them strictly positive. As is well known the fractional Laplacian is defined, up to a normalization constant (which we omit for brevity), by a singular integral

$$(-\Delta)^s u(x) \coloneqq \lim_{r \to 0^+} \int_{\mathbb{R}^n \setminus B(x,r)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy.$$

We have to mention that the interest in problems involving mixed operators has increased significantly in recent years. This is partly due to their ability to capture phenomena where both local and nonlocal effects coexist that arises naturally in models combining classical diffusion (governed by Brownian motion) and anomalous diffusion (described by Lévy-type processes). Such a combination is relevant in diverse applications, including materials with heterogeneous microstructure, financial models with both local volatility and jumps. In biological systems where individuals may move via both short- and long-range interactions these type of operators appear too. From a theoretical perspective, these problems pose interesting challenges due to the competing regularity and scaling behaviors of the local and nonlocal parts. Some non exhaustive list reference in the topic could be [7, 8, 1, 13, 11, 12, 14, 29, 31].

About the notion of critical exponents, it is well known that the critical exponent in the whole space associated to

$$F(u) = u^p,$$

where $F$ is a (possibly nonlinear) elliptic operator, local or nonlocal in nature, is defined by

$$p_*(F) \coloneqq \inf\{p \in \mathbb{R} \colon \text{there exists a positive solution solution to } F(u) = u^p \text{ in } \mathbb{R}^n\}.$$

It is also known by the community that Lane-Emden type equations at its critical exponent play a central role in the study of PDE's problems as illustrated the classical case $F = -\Delta$ which connects the Yamabe problem and sharp constants in Sobolev inequalities. See for instance [46, 45, 33].

On the other hand, the critical exponent in a bounded (smooth) domain is defined by

$$p^*(\Omega, F) \coloneqq \sup\{p \in \mathbb{R} \colon \text{ there exists a positive solution to } F(u) = u^p \text{ in } \Omega\}.$$

For the classical Laplacian $F = -\Delta$ and when $\Omega$ is star-shaped, it is well known that a form of *duality* appears, that is,

$$p^*(\Omega, -\Delta) = p_*(-\Delta) = p_1 \coloneqq \frac{n+2}{n-2},$$

see [39, 17, 19, 32]. In the radial case, this duality of the exponent (that is $p^*(B_R, -\Delta) = p_*(-\Delta)$) is clear, since a solution $u_\gamma(r)$, $r \in (0, +\infty)$ is a solution of an ODE with positive initial condition $u(0) = \gamma > 0$ that cross or not the $r$-axes. In other words, if $1 < p < p_1$ then there exists $r_0$ such $u_\gamma(r_0) = 0$, and if $p \geq p_1$ then $u_\gamma(r) > 0$ for all $r \in (0, +\infty)$. By scaling $\gamma$ can be an arbitrary value of $\mathbb{R}^+$. The Emden-Fowler transformation is a classical change of variable to study this types of equations and gives the radial duality described above.

Thus, it seems natural to find some domains $\Omega$ with non-trivial topology (not star-shaped) such that $p^*(\Omega, -\Delta) > p_1$. See for instance [22] and also the starting point in [5] at the critical exponent $p_1$. The case of the exterior domain is discussed for example in [20] and the case of half space in [25]. These types of domains are interesting for mixed operators but these cases fall outside the scope of this work in which we focus on the complete Euclidean space.

Returning to star-shaped domains or the whole space, similar results than the local ones, hold when considering the purely nonlocal operator $F = L_{0,1} = (-\Delta)^s$ (i.e. $a = 0$ and $b = 1$). In fact, if $\Omega$ is star-shaped then we also have duality of the exponents and

$$p^*(\Omega, (-\Delta)^s) = p_*((-\Delta)^s) = p_s := \frac{n+2s}{n-2s},$$

(see [40, 42, 19]). Variational methods guarantee existence of solutions in any regular bounded domain $\Omega$ for the subcritical range $1 < p < p_s$, when $\Omega$ is not necessarily star-shaped, see [42]. Other results posed in the pure nonlocal setting regarding critical can be found in, for example, [9], where a Hénon-type equation is discussed.

Some other equations that have sub and supercritical phenomenas similar to the problem (2) are for instance,

$$\Delta u + u^p + u^q = 0 \quad \text{in} \quad \mathbb{R}^n,\ 1 < p < p_1 < q,$$

studied in the seminal work of [34]. For that equation existence of positive solution with fast decay are showed in [6]. See also [18].

Another closely connected problem with sub and supercritical behavior is the Matukuna type problem of the form

$$\Delta u + K(r)u^p = 0, \text{ in } \mathbb{R}^n,$$

studied in [27] where fast decay solution exists for special $K$ having sub and super critical behavior. Moreover, the case where $K$ is a mixed Henon type with sub and supercritical behavior is studied in [2]. Fast decay solutions also exist.

Returning to the case of mixed operators, if $\Omega$ is a star-shaped domain and $a, b > 0$ it has been shown in Theorem 1.3 in [41]( see also example iv in Section 2) that $p^*(\Omega, L_{a,b}) = p_1$. This value of the critical exponent also appears in a Brezis–Nirenberg type result [12]. Using variational methods, existence results in bounded domains have been established for the subcritical range $1 < p < p_1$ in [36] even when $b < 0$.

This leads to the natural **conjecture** for star-shaped domains

$$p_*(L_{a,b}) = p^*(\Omega, L_{a,b}) = p_1,$$

that affirms that also the duality holds for the mixed operator.

The main objective of the present work is to establish that the **conjecture** is false and that, in fact, if $\Omega$ is a star-shaped domain it holds

$$p_*(L_{a,b}) < p^*(\Omega, L_{a,b}) = p_1,$$

providing the first example of an elliptic operator $L_{a,b}$ for which the duality between critical exponents fails. Before detailing the concrete unexpected result we are able to prove, we state a related conjecture that read as follows.

**Conjecture:** $p_*(L_{a,b}) = p_s$.

The proof of the previous conjecture need to be split into two parts.

i) If $1 < p < p_s$ one may expect that there is no solution to

$$L_{a,b}u = u^p \text{ in } \mathbb{R}^N. \tag{1}$$

We notice that if we assume some extra integrability conditions on the solutions at its gradient, this can be proved by applying the Pohozaev identity given in Remark 3.1 of [41].

ii) The existence of solutions to (1) for $p = p_s + \varepsilon$ could be proved following a similar (but not straightforward at all) perturbation-type arguments of the used in the present work. We think that a different kind of functional framework should be used and the *bubble* function associated to $(-\Delta)^s$ may play an important role in the bifurcation procedure.

Such critical exponents are tightly related to fundamental threshold phenomena, where the existence of positive solutions typically changes abruptly. Moreover, understanding the value of them for general operators $F$, including nonlocal or mixed ones like $-\Delta + (-\Delta)^s$, not only extends the classical theory but also sheds light on new regimes of nonlinearity, where local and nonlocal effects interact in nontrivial ways.

Coming back to our objective, as we mentioned, we want to establish the existence of radial solutions to

$$\begin{cases} -\Delta u + (-\Delta)^s u = n(n-2)u^{p_\varepsilon} & \text{in } \mathbb{R}^n,\ p_\varepsilon := p_1 - \varepsilon,\ n \geq 3, \\ u > 0 & \text{in } \mathbb{R}^n \end{cases} \tag{2}$$

where for simplicity, without loss of generality we have taken

$$a = b = \frac{1}{n(n-2)}$$

in $L_{a,b}$, and $\varepsilon > 0$ will be a small parameter. It is clear that $u_\varepsilon$ is a solution of (2) if and only if

$$u_{\delta,\varepsilon}(x) := \delta^{\frac{2}{p_\varepsilon - 1}} u_\varepsilon(\delta x), \text{ with } \delta > 0,$$

is a solution of

$$\begin{cases} -\Delta u + \delta^{2(1-s)}(-\Delta)^s u = n(n-2)u^{p_\varepsilon} & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{3}$$

in a suitable energy space (see (5) below). Therefore, since it is well known (see [45]) that the local problem

$$\begin{cases} -\Delta u = n(n-2)u^{p_1} & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n, \end{cases}$$

has the family of Talenti functions (scale by the initial condition)

$$U_\delta(x) := \delta^{\frac{n-2}{2}} U\left(\frac{x}{\delta}\right),\ U(x) := \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}},\ \delta > 0, y \in \mathbb{R}^n, \tag{4}$$

as unique solutions, one of our goals is to show the following.

**Theorem 1.1.** *Let $n \geq 3$ and $\varepsilon > 0$. There exists $\delta(\varepsilon)$ such that the problem (3) has a radial solution of the form*

$$z_\varepsilon(x) = U(x) + \phi_\varepsilon(x),$$

*where $\phi_\varepsilon \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $U$ is given in (4). In addition, by rescaling back, there exists a solution of (2) that concentrates at the origin.*

To prove it, we use a Lyapunov–Schmidt reduction method that has first used in [30] in the context of partial differential equations and has become a very active area in PDEs since then. For a review of some other perturbation methods in $\mathbb{R}^n$ see the book [4] and also [18, 2]. In our particular case we follow the Lyapunov–Schmidt reduction in the spirit of [37] wherethe difficulties that the nonlocal operator introduces appear. The main idea of this approach is to solve the problem in the orthogonal of the kernel of the linearized operator denoted by $K$ with some parameters (finite reduction) and then find parameters so that there is also a solution to the full problem. The parameters are chosen so the problem is always in the orthogonal space of the Kernel $K$ so is a solution to the problem in all space (not only in the orthogonal of the $K$).

The paper is organized as follows. In Section 2 we introduce the framework and the notation we will use along the work. Moreover we prove several auxiliar lemmas that will be needed to establish the proof of Theorem 1.1 that is detailed in Section 3. We want to highlight the Lemma 2.8, a key point in our approach, which establishes that the $H^s(\mathbb{R}^n)$ semi-inner product of the bubble solution with the unique element of the Kernel $K$ has a negative sign. From here we can find parameters to guarantee that the projected solution is a full solution.

## 2. Problem Setting and Notations

Given $s \in (0,1]$, $\mathcal{D}^{s,2}(\mathbb{R}^n)$ denotes the clausure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{s,2}^2 := \begin{cases} \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dy\,dx = [u]_{H^s} & \text{if } s \in (0,1),\\ \displaystyle\int_{\mathbb{R}^n} |\nabla u|^2\,dx = [u]_{H^1} & \text{if } s = 1.\end{cases}$$

It is important to note that when it is equipped with the inner product

$$\langle u, v\rangle_s := \begin{cases} \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dy\,dx & \text{if } s \in (0,1),\\ \displaystyle\int_{\mathbb{R}^n} \nabla u \nabla v\,dx & \text{if } s = 1,\end{cases}$$

the space $\mathcal{D}^{s,2}(\mathbb{R}^n)$ becomes a Hilbert space. In the case $s = 1$, for notational convenience we write $\langle\cdot,\cdot\rangle$ in place of $\langle\cdot,\cdot\rangle_1$. We set

$$\mathcal{X} := \mathcal{D}^{1,2}(\mathbb{R}^n) \cap L^{\frac{2n}{n+2}}(\mathbb{R}^n), \tag{5}$$

with the norm

$$\|u\| := \max\left\{\|u\|_{1,2}, \|u\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}\right\}, \tag{6}$$

By the Nirenberg-Sobolev inequality and interpolation, we have the inclusion $\mathcal{X} \subset L^q(\mathbb{R}^n)$ with $q \in [2n/(n+2), 2n/(n-2)]$. In particular, since $2 \in [2n/(n+2), 2n/(n-2)]$, it follows that $\mathcal{X} \subset L^2(\mathbb{R}^n)$. Consequently, by [23, Proposition 2.1], we deduce

$$\mathcal{X} \subset \mathcal{D}^{s,2}(\mathbb{R}^n),$$

for any $s \in (0,1)$.

**Definition 2.1.** Given $s \in (0,1)$, and $\delta \geq 0$, we define the operator

$$\mathcal{I}_\delta \colon L^{\frac{2n}{n+2}}(\mathbb{R}^n) \to \mathcal{D}^{1,2}(\mathbb{R}^n),$$

by setting

$$\mathcal{I}_\delta(h) = u,$$

if and only if $u$ is a weak solution of

$$(-\Delta)u + \delta^{2(1-s)}(-\Delta)^s u + \delta^{2(1-s)} u = h \quad \text{in} \quad \mathbb{R}^n. \tag{7}$$

That is, $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ satisfies

$$\langle u, v\rangle + \delta^{2(1-s)}\langle u, v\rangle_s + \delta^{2(1-s)} \int_{\mathbb{R}^n} uv\, dx = \int_{\mathbb{R}^n} hv\, dx,$$

for any $v \in \mathcal{D}^{1,2}(\mathbb{R}^n)$.

By [10, Lemma 6.9 and Corollary 6.7], [43, Section 3.3 in Chapter 5] and [24, Lemma 3.3], we have the following result.

**Lemma 2.1.** *Let $\delta \geq 0$ and $h \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$. If $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a weak solution of (7) then $u \in W^{2,\frac{2n}{n+2}}(\mathbb{R}^n)$ and there exists $C = C(n,\delta) > 0$ such that*

$$\|u\|_{W^{2,\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}.$$

**Remark 2.1.** As a consequence of Lemma 2.1, the operator $\mathcal{I}_\delta$ maps into the space $\mathcal{X}$, that is

$$\mathcal{I}_\delta \colon L^{\frac{2n}{n+2}}(\mathbb{R}^n) \to \mathcal{X}.$$

Moreover $\mathcal{I}_\delta$ is continuous. Specifically, there exists a constant $C = C(n,\delta)$ such that

$$\|\mathcal{I}_\delta(h)\| \leq C\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)},$$

for all $h \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$.

Clearly $u$ is a weak solution of (3) if and only if

$$u = \mathcal{I}_\delta(g_{\varepsilon,\delta}(u)) \text{ and } u \in \mathcal{X}. \tag{8}$$

where

$$g_{\varepsilon,\delta}(t) := f_\varepsilon(t) + \delta^{2(1-s)}t, \text{ with } f_\varepsilon(t) = n(n-2)(t_+)^{p_\varepsilon}, \tag{9}$$

here $t_+$ denotes the positive part of $t$. We also notice that

**Lemma 2.2.** *If $u \in \mathcal{X}$ is a non-trivial weak solution of (8) (i.e. of (3)) then $u > 0$ a.e. in $\mathbb{R}^n$.*

*Proof.* Let $u \in \mathcal{X}$ be a non-trivial solution of (8). Testing (8) with $u_-(x) := \max\{-u(x), 0\} \geq 0$, $(u = u_+ - u_-,\ u_+ = \max\{u, 0\}$ ), we clearly get

$$\begin{aligned} 0 &= -\|\nabla u_-\|_{1,2}^2 + \delta^{2(1-s)}\langle u_+ - u_-, u_-\rangle_s \\ &= -\|\nabla u_-\|_{1,2}^2 + \delta^{2(1-s)}\left(\langle u_+, u_-\rangle_s - [u_-]_{s,2}^2\right), \end{aligned} \tag{10}$$

so that $\langle u_+, u_-\rangle_s \geq 0$ with in fact implies that $\langle u_+, u_-\rangle_s = 0$. Thus, from (10) we obtain that $u_- \equiv C$, $C \in \mathbb{R}$ that, since $u \in \mathcal{X}$, implies $C = 0$ as wanted. To get the strictly positivity of the solution, since we are working in weak sense, using a Logarithmic Lemma (see [26]) and following the ideas developed in [21, Lemma 3.3] we conclude.

□

Our next goal is to derive estimates depending on the parameters $\varepsilon$ and $\delta$. For this purpose, we need the following $L^q-$estimate for the fractional Laplacian.

**Lemma 2.3.** *For any $u \in W^{2,q}(\mathbb{R}^n)$, it holds that $(-\Delta)^s u \in L^q(\mathbb{R}^n)$ and*

$$\|(-\Delta)^s u\|_{L^q(\mathbb{R}^n)} \leq C\|u\|_{W^{2,q}(\mathbb{R}^n)},$$

*for some positive constant $C$ depending only on $n, s$ and $q$.*

*Proof.* The proof follows ideas from [44, Subsection 4.1.1].

Let $u \in W^{2,q}(\mathbb{R}^n)$. By the Minkowski inequality, we estimate

$$\begin{aligned} &\left(\int_{\mathbb{R}^n}\left|\int_{|y|<1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy\right|^q dx\right)^{\frac{1}{q}} \leq \\ &\leq \left(\int_{\mathbb{R}^n}\left(\int_{|y|<1}\int_0^1\int_0^\theta |D^2u(x+\eta y)|d\eta d\theta\frac{dy}{|y|^{n+2(s-1)}}\right)^q dx\right)^{\frac{1}{q}} \\ &\leq \int_{|y|<1}\left(\int_{\mathbb{R}^n}\int_0^1\int_0^\theta |D^2u(x+\eta y)|^q d\eta d\theta\, dx\right)^{\frac{1}{q}}\frac{dy}{|y|^{n+2(s-1)}} \\ &\leq C(n,s)\|u\|_{W^{2,q}(\mathbb{R}^n)}. \end{aligned}$$

For the remainder of the proof, that is for the integral estimate when $|y| \geq 1$ we refer to the estimate of $III$ in the proof of Lemma 4.2 in [44]. This completes the proof. □

**Lemma 2.4.** *Let $s \in (0,1)$, $0 < \varepsilon, \delta < 1$. Let $\phi \in \mathcal{X}$, and define*

$$w_{\varepsilon,\delta} = \mathcal{I}_\delta(g_{\varepsilon,\delta}(U + \phi)),$$

*where $g_{\varepsilon,\delta}$ is given in (9) and $U$ is defined in (4) Then, for $\varepsilon$, and $\delta$ small enough, there exists a constant $C > 0$ independent of $\varepsilon$, and $\delta$ such that*

$$\|w_{\varepsilon,\delta} - U\| \le C \left( \|\phi\|^{p_1-\varepsilon} + \|\phi\| + \varepsilon + \delta^{2(1-s)} \right),$$

*for any $\phi \in \mathcal{X}$ where $\|\cdot\|$ was given in (6).*

*Proof.* Let $z := w_{\varepsilon,\delta} - U$. Then $z$ satisfies the equation

$$-\Delta z + \delta^{2(1-s)}(-\Delta)^s z + \delta^{2(1-s)} z = f_\varepsilon(U+\phi) - f_0(U) + \delta^{2(1-s)}\phi - \delta^{2(1-s)}(-\Delta)^s U.$$

Firstly we estimate each terms of the right hand side

$$H := f_\varepsilon(U+\phi) - f_0(U) + \delta^{2(1-s)}\phi - \delta^{2(1-s)}(-\Delta)^s U.$$

in the space $L^{\frac{2n}{n+2}}(\mathbb{R}^n)$. For that we observe that by [37, Remarks 2.21 and 2.22], for sufficiently small $\varepsilon$, and $\delta$ there exists a positive constant $C$, independent of $\varepsilon$, and $\delta$, such that

$$\begin{aligned}
\|f_\varepsilon(U+\phi) - f_0(U)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} &= \|f_\varepsilon(U+\phi) - f_\varepsilon(U) - f'_\varepsilon(U)\phi\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\
&\quad + \|f_\varepsilon(U) - f_0(U)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} + \|f'_\varepsilon(U)\phi\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\
&\le C \left( \|\phi\|^{p_1-\varepsilon} + \varepsilon + \|\phi\| \right).
\end{aligned}$$

Moreover, since from [28, Page 259], we know that

$$(-\Delta)^s U \in L^{\frac{2n}{n+2}}(\mathbb{R}^n), \tag{11}$$

we clearly get that

$$\|H\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \le C \left( \|\phi\|^{p_1-\varepsilon} + \|\phi\| + \varepsilon + \delta^{2(1-s)} \right), \tag{12}$$

for some positive constant $C$, independent of $\varepsilon$ and $\delta$, provided both are sufficiently small. Then, by Lemma 2.1 we get that $z \in W^{2,\frac{2n}{n+2}}(\mathbb{R}^n)$ and

$$\|z\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \le \|z\|_{W^{2,\frac{2n}{n+2}}(\mathbb{R}^n)} \le C \left( \|\phi\|^{p_1-\varepsilon} + \|\phi\| + \varepsilon + \delta^{2(1-s)} \right),$$

where $C$ is a positive constant independent of $\varepsilon$ and $\delta$, provided both are sufficiently small.

On the other hand, by Hölder and Nirenberg-Sobolev inequalities, by (12) we also obtain

$$\begin{aligned}
\|z\|_{1,2}^2 + \delta^{2(1-s)}\langle z, z\rangle_s + \delta^{2(1-s)}\|z\|_{L^2(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} Hz\,dx \\
&\le \|H\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \|z\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\
&\le C \left( \|\phi\|^{p_1-\varepsilon} + \|\phi\| + \varepsilon + \delta^{2(1-s)} \right) \|z\|_{1,2}.
\end{aligned}$$

Therefore

$$\|z\|_{1,2} \le C \left( \|\phi\|^{p_1-\varepsilon} + \|\phi\| + \varepsilon + \delta^{2(1-s)} \right),$$

which concludes the proof. □

We introduce the following fundamental definition.

**Definition 2.2.** Let $\delta \geq 0$. The operator $L_\delta \colon \mathcal{X} \to \mathcal{X}$ is given by

$$L_\delta(u) \coloneqq u - \mathcal{I}_\delta(g'_{0,\delta}(U)u). \tag{13}$$

By [16, Lemma 2.8] we notice that $L_0$ is a self-adjoint operator and is a compact perturbation of the identity on $\mathcal{X}$. We show now that the family of operators $L_\delta$ converges to $L_0$ as $\delta \to 0$.

**Lemma 2.5.** *For every $u \in \mathcal{X}$ there exists $C > 0$, independent of $\delta$, such that*

$$\|L_\delta(u) - L_0(u)\| \leq C\delta^{2(1-s)},$$

*provided as $\delta$ is small enough.*

*Proof.* Let $u \in \mathcal{X}$, and $w_\delta \coloneqq L_\delta(u)$, $\delta \geq 0$. Then, by definition

$$-\Delta(u - w_0) = n(n+2)U^{\frac{4}{n-2}}u. \tag{14}$$

Since $0 < U \in L^\infty(\mathbb{R}^n)$, and $u \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$, it follows from Lemma 2.1 that $u - w_0 \in W^{2,\frac{2n}{n+2}}(\mathbb{R}^n)$. Moreover, by Lemma 2.3, we also have that $(-\Delta)^s(u - w_0) \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$.

Let now $z_\delta \coloneqq w_0 - w_\delta$ and $\widetilde{H} \coloneqq (-\Delta)^s(w_0 - u) + w_0 \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$. Since $w_\delta = u - v$ with

$$(-\Delta)v + \delta^{2(1-s)}(-\Delta)^s v + \delta^{2(1-s)}v = n(n+2)U^{\frac{4}{n-2}}u + \delta^{2(1-s)u},$$

by (14) we get that $z_\delta$ satisfies the equation

$$-\Delta z_\delta + \delta^{2(1-s)}(-\Delta)^s z_\delta + \delta^{2(1-s)}z_\delta = \delta^{2(1-s)}\widetilde{H}. \tag{15}$$

By Lemma 2.1, we deduce that $z_\delta \in W^{2,\frac{2n}{n+2}}(\mathbb{R}^n)$ and

$$\|z_\delta\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq \|z_\delta\|_{W^{2,\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\delta^{2(1-s)},$$

where $C > 0$ is independent of $\delta$.

Testing (15) against $z_\delta$ and using Nirenberg-Sobolev inequality, we get

$$\|z_\delta\|_{1,2}^2 \leq \delta^{2(1-s)}\|\widetilde{H}\|_{L^{\frac{2n}{n+2}}(\mathbb{R})}\|z_\delta\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\delta^{2(1-s)}\|z_\delta\|_{1,2},$$

where $C$ is a positive constant independent of $\delta$. This completes the proof. □

From now on, we restrict our analysis to the subspace

$$\mathcal{R} \coloneqq \{u \in \mathcal{X} \colon u \text{ is radial}\}.$$

of radial functions. We have the following result whose proof we outline for completeness.

**Lemma 2.6.** *Let $\delta \geq 0$. If $u \in \mathcal{R}$ then $L_\delta(u) \in \mathcal{R}$, that is $L_\delta\big|_{\mathcal{R}} \colon \mathcal{R} \to \mathcal{R}$. Moroever,*

$$\operatorname{Ker} L_0\big|_{\mathcal{R}} = \operatorname{span}\{\psi\} \tag{16}$$

*where*

$$\psi(x) \coloneqq x \cdot \nabla U + \frac{n-2}{2} U = \frac{n-2}{2} \frac{1-|x|^2}{(1+|x|^2)^{n/2}}. \tag{17}$$

*Proof.* The proof of $L_\delta\big|_{\mathcal{R}} \colon \mathcal{R} \to \mathcal{R}$, is given in [24]. To show that (16) holds, see [3, Lemma 3.12] or [38, Lemma 4.2]. □

**Remark 2.2.** If $n \geq 3$ then $\psi \in H^1(\mathbb{R}^n)$ and therefore $\psi \in H^s(\mathbb{R}^n)$.

We define now

$$K \coloneqq \{v \in \mathcal{R} \colon \langle v, \psi \rangle_{1,2} = 0\},$$

and the continuous projection

$$\Pi \colon \mathcal{R} \to K,$$

Moreover, for every $\delta \geq 0$ we define the operator $\widetilde{L}_\delta \colon K \to K$ as

$$\widetilde{L}_\delta(u) \coloneqq \Pi(L_\delta\big|_{\mathcal{R}}(u)).$$

where $L_\delta$ was given in (13). According to [37], the operator $\widetilde{L}_0$ is invertible and $\widetilde{L}_0^{-1}$ is continuous. Moreover we can shows that $\widetilde{L}_\delta$ is invertible and $\widetilde{L}_\delta^{-1}$ is continuous, as the following result reads.

**Lemma 2.7.** *There exists $\delta_0 > 0$ such that $\widetilde{L}_\delta$ is invertible and $\widetilde{L}_\delta^{-1}$ is continuous, for every $\delta \in (0, \delta_0)$ .*

*Proof.* Using Lemma 2.5, is clear that $\widetilde{L}_\delta(u) = \widetilde{L}_0(u) + R_\delta(u)$ where

$$\|R_\delta\| \leq C\delta^{2(1-s)}, \tag{18}$$

with $C$ being a positive constant independent of $\delta$. Then

$$\widetilde{L}_\delta = \widetilde{L}_0(Id + \widetilde{L}_0^{-1} R_\delta).$$

Thus, by (18), for sufficiently small $\delta$, the operator $\widetilde{L}_\delta$ is invertible and

$$\widetilde{L}_\delta^{-1} = (Id + \widetilde{L}_0^{-1} R_\delta)^{-1} \widetilde{L}_0^{-1}.$$

□

We conclude this section with the following lemma that will be essential for proving our main result.

**Lemma 2.8.** *Let $n > 2$ and $s \in (0,1)$. Then*

$$\int_{\mathbb{R}^n} \log(U) U^{p_1} \psi \, dx > 0, \tag{19}$$

*and*

$$\langle U, \psi \rangle_s < 0, \tag{20}$$

*where $\psi$ was given in (17).*

*Proof.* To prove (19), see the proof of Lemma 2.19 in [37].

For (20), we use [28, (6.5)], where it is shown that

$$(-\Delta)^s U(x) = C(n,s) F\left(\frac{n+2s}{2}, \frac{n-2(1-s)}{2}, \frac{n}{2}, -|x|^2\right),$$

where $C(n,s)$ is a positive constant and $F$ denotes the Gaussian hypergeometric function. Then,

$$\begin{aligned}\langle U, \psi\rangle_s &= C(n,s)\frac{n-2}{2}\int_{\mathbb{R}^n} F\left(\frac{n+2s}{2}, \frac{n-2(1-s)}{2}, \frac{n}{2}, -|x|^2\right)\psi(x)\,dx \\ &= K(n,s)\int_0^\infty F\left(\frac{n+2s}{2}, \frac{n-2(1-s)}{2}, \frac{n}{2}, -\rho^2\right)\frac{1-\rho^2}{(1+\rho^2)^{n/2}}\rho^{n-1}\,d\rho \\ &= K(n,s)(I_1+I_2),\end{aligned} \tag{21}$$

where $K(n,s)$ is a positive constant, and we define

$$I_1 := \int_0^1 F\left(\frac{n+2s}{2}, \frac{n-2(1-s)}{2}, \frac{n}{2}, -\rho^2\right)\frac{1-\rho^2}{(1+\rho^2)^{n/2}}\rho^{n-1}\,d\rho,$$

and

$$I_2 := \int_1^\infty F\left(\frac{n+2s}{2}, \frac{n-2(1-s)}{2}, \frac{n}{2}, -\rho^2\right)\frac{1-\rho^2}{(1+\rho^2)^{n/2}}\rho^{n-1}\,d\rho.$$

By making the substitution $\rho = \tau^{-1}$, we have

$$\begin{aligned}I_2 &= \int_0^1 F\left(\frac{n+2s}{2}, \frac{n-2(1-s)}{2}, \frac{n}{2}, -\tau^{-2}\right)\frac{1-\tau^{-2}}{(1+\tau^{-2})^{n/2}}\tau^{1-n}\frac{d\tau}{\tau^2} \\ &= -\int_0^1 F\left(\frac{n+2s}{2}, \frac{n-2(1-s)}{2}, \frac{n}{2}, -\tau^{-2}\right)\frac{1-\tau^2}{(1+\tau^2)^{n/2}}\tau^{-3}\,d\tau.\end{aligned}$$

Then,

$$I_1 + I_2 = \int_0^1 H(\rho)\frac{1-\rho^2}{(1+\rho^2)^{n/2}}\,d\rho,$$

where

$$\begin{aligned}H(\rho) = F&\left(\frac{n+2s}{2}, \frac{n-2(1-s)}{2}, \frac{n}{2}, -\rho^2\right)\rho^{n-1} \\ &- F\left(\frac{n+2s}{2}, \frac{n-2(1-s)}{2}, \frac{n}{2}, -\rho^{-2}\right)\rho^{-3}.\end{aligned}$$

If

$$H(\rho) < 0 \quad \text{in } (0,1), \tag{22}$$

then

$$I_1 + I_2 < 0.$$

Thus, from (21), we conclude that (20) holds. To complete the proof, it remains to verify that (22) holds.

By [35, Section 2.5, page 54], we have

$$\frac{\Gamma\left(\frac{n-2(1-s)}{2}\right)\Gamma(1-s)}{\Gamma\left(\frac{n}{2}\right)}H(\rho) =$$

$$= \int_0^1 \frac{t^{(n-2(2-s))/2}}{(1-t)^s}\left[(1+\rho^2 t)^{-(n+2s)/2}\rho^{n-1} - (\rho^2+t)^{-(n+2s)/2}\rho^{n+2s-3}\right] dt$$

$$= \int_0^1 \frac{t^{(n-2(2-s))/2}}{(1-t)^s}\frac{\rho^{n+2s-3}}{(\rho^2+t)^{(n+2s)/2}}\left[\left(\frac{\rho^2+t}{1+\rho^2 t}\right)^{(n+2s)/2}\rho^{2(1-s)} - 1\right] dt.$$

Since

$$\left(\frac{\rho^2+t}{1+\rho^2 t}\right)^{(n+2s)/2}\rho^{2(1-s)} - 1 < 0 \quad \forall t, \rho \in (0,1),$$

we conclude that

$$H(\rho) < 0 \quad \forall \rho \in (0,1).$$

□

## 3. EXISTENCE

In this section, we demonstrate the existence of a radial function $\phi_{\varepsilon,\delta}$ such that $U + \phi_{\varepsilon,\delta}$ satisfies equation (8).

We begin by noting that if $U + \phi_{\varepsilon,\delta}$ is a radial solution to (8), it must also satisfy

$$\Pi(U + \phi_{\varepsilon,\delta} - \mathcal{I}_\delta(g_{\varepsilon,\delta}(U + \phi_{\varepsilon,\delta}))) = 0, \tag{23}$$

where $g_{\varepsilon,\delta}$ was given in (9). Then, our initial objective is to prove the existence of a function $\phi_{\varepsilon,\delta}$ such that $U + \phi_{\varepsilon,\delta}$ solves (23). To achieve this, we define the operator $T_{\varepsilon,\delta}\colon K \to K$ by

$$T_{\varepsilon,\delta}(\phi) := \widetilde{L}_\delta^{-1}\Big(\Pi\Big(\mathcal{I}_\delta\big(f_\varepsilon(U+\phi) - f_0(U) - f_0'(U)\phi - \delta^{2(1-s)}(-\Delta)^s U\big)\Big)\Big).$$

Taking into account the decay of (4), by using [15, Lemma 2.1] with $\alpha = N - 1$ we get that $U \in \mathcal{D}^{1,2}(\mathbb{R}^n)$. Thus, by (11) is clear that $(-\Delta)^s U \in \mathcal{X}$ so $T_{\varepsilon,\delta}$ is well defined.

Observe that if $\phi_{\varepsilon,\delta}$ is a fixed point of $T_{\varepsilon,\delta}$, then $U + \phi_{\varepsilon,\delta}$ is a solution of equation (23). Thus, our first task reduces to showing that $T_{\varepsilon,\delta}$ possesses a fixed point.

Once we establish the existence of a fixed point $\phi_{\varepsilon,\delta}$ for $T_{\varepsilon,\delta}$, our next objective will be to demonstrate that, for appropriate values of $\delta$ and $\varepsilon$, we have

$$\langle U + \phi_{\varepsilon,\delta} - \mathcal{I}_\delta(g_{\varepsilon,\delta}(U + \phi_{\varepsilon,\delta})), \psi\rangle = 0. \tag{24}$$

Finally by (23) and (24), we will conclude that $U + \phi_{\varepsilon,\delta}$ satisfies equation (8) for appropriate values of $\delta$ and $\varepsilon$.

To prove the existence of a fixed point, we first get the following

**Lemma 3.1.** *Let $\alpha \in (0,1)$ and $\phi \in B^{\alpha}_{\varepsilon,\delta} := \{\phi \in K : \|\phi\| \leq (\varepsilon + \delta^{2(1-s)})^{\alpha}\}$. Then*

$$\|T_{\delta,\epsilon}(\phi)\| \leq (\varepsilon + \delta^{2(1-s)})^{\alpha},$$

*for $\varepsilon, \delta$ small enough. That is, $T_{\delta,\epsilon}|_{B^{\alpha}_{\varepsilon,\delta}} : B^{\alpha}_{\varepsilon,\delta} \to B^{\alpha}_{\varepsilon,\delta}$.*

*Proof.* Let $\alpha \in (0,1)$. Given $\phi \in K$ , by Remark 2.1 and Lemma 2.6 is clear that

$$\begin{aligned}
\|T_{\delta,\epsilon}(\phi)\| &\leq C\|\mathcal{I}_{\delta}(f_{\varepsilon}(U+\phi) - f_0(U) - f_0'(U)\phi - \delta^{2(1-s)}(-\Delta)^s U))\| \\
&\leq C\|\mathcal{I}_{\delta}(f_{\varepsilon}(U+\phi) - f_{\varepsilon}(U) - f_{\varepsilon}'(U)\phi)\| + C\|\mathcal{I}_{\delta}(f_{\varepsilon}(U) - f_0(U))\| \\
&+ C\|\mathcal{I}_{\delta}(f_{\varepsilon}'(U)\phi - f_0'(U)\phi)\| + \delta^{2(1-s)}C\|\mathcal{I}_{\delta}((-\Delta)^s U))\| \\
&\leq C\|\mathcal{I}_{\delta}(f_{\varepsilon}(U+\phi) - f_{\varepsilon}(U) - f_{\varepsilon}'(U)\phi\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\
&+ C\|f_{\varepsilon}(U) - f_0(U)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} + C\|f_{\varepsilon}'(U)\phi - f_0'(U)\phi\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\
&+ \delta^{2(1-s)}C\|(-\Delta)^s U)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}.
\end{aligned}$$

Then, by H$\acute{}$older inequality and using [37, Remark 2.1 and Remark 2.22], we get

$$\|T_{\delta,\epsilon}(\phi)\| \leq C\left(\|\phi\|^{p_1-\varepsilon} + \varepsilon + \varepsilon\|\phi\| + \delta^{2(1-s)}\right),$$

for $\varepsilon$ small enough. Imposing now the condition $\|\phi\| \leq (\varepsilon + \delta^{2(1-s)})^{\alpha}$ we conclude

$$\begin{aligned}
\|T_{\delta,\epsilon}(\phi)\| &\leq C\left((\varepsilon + \delta^{2(1-s)})^{\alpha(p_1-\varepsilon)} + (\varepsilon + \delta^{2(1-s)})^{\alpha+1} + \varepsilon + \delta^{2(1-s)}\right) \\
&\leq (\varepsilon + \delta^{2(1-s)})^{\alpha},
\end{aligned}$$

due to $p_1 - \varepsilon > 1$ and the fact that $\alpha \in (0,1)$. □

Our second step is to prove the next.

**Lemma 3.2.** *For $\varepsilon$ small enough we have that $T_{\delta,\epsilon}|_{B^{\alpha}_{\varepsilon,\delta}}$ is a contraction mapping.*

*Proof.* Let $\phi_1, \phi_2 \in B^{\alpha}_{\varepsilon,\delta}$. Then

$$\begin{aligned}
\|T_{\delta,\epsilon}(\phi_1) - T_{\delta,\epsilon}(\phi_2)\| &\leq C\|\mathcal{I}_{\delta}(f_{\varepsilon}(U+\phi_1) - f_{\varepsilon}(U+\phi_2) - f_0'(U)(\phi_1 - \phi_2))\| \\
&\leq C\|\mathcal{I}_{\delta}(f_{\varepsilon}(U+\phi_1) - f_{\varepsilon}(U+\phi_2) - f_{\varepsilon}'(U+\phi_2)(\phi_1 - \phi_2)))\| \\
&+ C\|\mathcal{I}_{\delta}((f_{\varepsilon}'(U+\phi_2) - f_{\varepsilon}'(U))(\phi_1 - \phi_2))\| \\
&+ C\|\mathcal{I}_{\delta}((f_{\varepsilon}'(U) - f_0'(U))(\phi_1 - \phi_2))\|.
\end{aligned}$$

Now by Remark 2.1 and [37, Remark 2.1 and Remark 2.2.], we get

$$\begin{aligned}
&\|T_{\delta,\epsilon}(\phi_1) - T_{\delta,\epsilon}(\phi_2)\| \\
&\leq C\left(\|\phi_1 - \phi_2\|^{p_1-\varepsilon} + \|\phi_2\|^{p_1-1-\varepsilon}\|\phi_1 - \phi_2\| + \varepsilon\|\phi_1 - \phi_2\|\right) \\
&\leq D\|\phi_1 - \phi_2\|,
\end{aligned}$$

for some positive constant $D < 1$ due to $p_1 - \varepsilon > 1$. □

Then, by Lemmas 3.1 and 3.2 we can conclude the existence of a fix point. That is.

**Lemma 3.3.** *For $\varepsilon$ and $\delta$ small enough there exists $\phi_{\varepsilon,\delta} \in B^{\alpha}_{\varepsilon,\delta}$ such that*

$$T_{\delta,\epsilon}(\phi_{\varepsilon,\delta}) = \phi_{\varepsilon,\delta}.$$

As we mentioned before, our final goal is to show that for appropriate values of $\delta$ and $\varepsilon$, we have (24). From now on, we take $\varepsilon$ such that $p_1 - \varepsilon > 1$, (i.e. $\varepsilon < \frac{4}{n-2}$), $\alpha > \frac{1}{p_1-\varepsilon}$ and we consider $\phi_{\varepsilon,\delta} \in B^{\alpha}_{\varepsilon,\delta}$.

**Lemma 3.4.** *For $0 < \varepsilon, \delta < 1$ small enough we get*

$$\langle U + \phi_{\varepsilon,\delta} - \mathcal{I}_\delta(g_{\varepsilon,\delta}(U + \phi_{\varepsilon,\delta})), \psi\rangle = \varepsilon \int_{\mathbb{R}^n} \log(U) U^{p_1} \psi \, dx + \delta^{2(1-s)} \langle U, \psi\rangle_s$$
$$+ o(\varepsilon + \delta^{2(1-s)}).$$

*Proof.* Let

$$W_{\varepsilon,\delta} \coloneqq \mathcal{I}_\delta(g_{\varepsilon,\delta}(U + \phi_{\varepsilon,\delta})).$$

Then, using the equations satisfy by $U$ and $\psi$, we get

$$\begin{aligned}
&\langle U + \phi_{\varepsilon,\delta} - \mathcal{I}_\delta(g_{\varepsilon,\delta}(U + \phi_{\varepsilon,\delta})), \psi\rangle = \langle U + \phi_{\varepsilon,\delta} - W_{\varepsilon,\delta}, \psi\rangle \\
&= \langle U, \psi\rangle + \langle \phi_{\varepsilon,\delta}, \psi\rangle - \langle W_{\varepsilon,\delta}, \psi\rangle \\
&= \int_{\mathbb{R}^n} \big(f_0(U) + f_0'(U)\phi_{\varepsilon,\delta} - f_\varepsilon(U + \phi_{\varepsilon,\delta})\big) \psi \, dx \\
&\qquad + \delta^{2(1-s)} \int_{\mathbb{R}^n} (W_{\varepsilon,\delta} - U)\psi \, dx + \delta^{2(1-s)} \langle W_{\varepsilon,\delta}, \psi\rangle_s - \delta^{2(1-s)} \int_{\mathbb{R}^n} \phi_{\varepsilon,\delta}\psi \, dx \\
&= \int_{\mathbb{R}^n} \big(f_0(U) + f_0'(U)\phi_{\varepsilon,\delta} - f_\varepsilon(U + \phi_{\varepsilon,\delta})\big) \psi \, dx + \delta^{2(1-s)} \int_{\mathbb{R}^n} (W_{\varepsilon,\delta} - U)\psi \, dx \\
&\qquad + \delta^{2(1-s)} \langle W_{\varepsilon,\delta} - U, \psi\rangle_s + \delta^{2(1-s)} \langle U, \psi\rangle_s - \delta^{2(1-s)} \int_{\mathbb{R}^n} \phi_{\varepsilon,\delta}\psi \, dx.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big(f_0(U) + f_0'(U)\phi_{\varepsilon,\delta} - f_\varepsilon(U + \phi_{\varepsilon,\delta})\big)\psi \, dx = \\
&\qquad = \int_{\mathbb{R}^n} \big(f_\varepsilon(U) + f_\varepsilon'(U)\phi_{\varepsilon,\delta} - f_\varepsilon(U + \phi_{\varepsilon,\delta})\big) \psi \, dx \\
&\qquad + \int_{\mathbb{R}^n} (f_0(U) - f_\varepsilon(U)) \psi \, dx + \int_{\mathbb{R}^n} \big(f_0'(U) - f_\varepsilon'(U)\big) \psi \, dx.
\end{aligned}$$

Following the computations done in [37, (3.20)–(3.23)], using that $\alpha > \frac{1}{p_1-\varepsilon}$ and $(p_1 - \varepsilon)\frac{2n}{n+2} \in \left(\frac{2n}{n+2}, \frac{2n}{n-2}\right)$ for $\varepsilon$ sufficiently small, we get

$$\int_{\mathbb{R}^n} \big(f_0(U) + f_0'(U)\phi_{\varepsilon,\delta} - f_\varepsilon(U + \phi_{\varepsilon,\delta})\big)\psi \, dx = \varepsilon \int_{\mathbb{R}^n} \log(U) U^{p_1} \psi \, dx + o(\varepsilon + \delta^{2(1-s)}).$$

On the other hand, since $p_1 - \varepsilon > 1$, applying Lemma 2.4 and Remark 2.2, we get

$$\begin{aligned}
&\left|\delta^{2(1-s)}\int_{\mathbb{R}^n}(w_{\varepsilon,\delta}-U)\psi\,dx+\delta^{2(1-s)}\langle w_{\varepsilon,\delta}-U,\psi\rangle_s-\delta^{2(1-s)}\int_{\mathbb{R}^n}\phi_{\varepsilon,\delta}\psi\,dx\right|\\
&\le C\left(\delta^{2(1-s)}\|w_{\varepsilon,\delta}-U\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}+\delta^{2(1-s)}\|w_{\varepsilon,\delta}-U\|_{s,2}+\delta^{2(1-s)}\|\phi_{\varepsilon,\delta}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}\right)\\
&\le C\delta^{2(1-s)}\left(\|\phi_{\varepsilon,\delta}\|^{p_1-\varepsilon}+\|\phi_{\varepsilon,\delta}\|+\varepsilon+\delta^{2(1-s)}\right)\\
&\le C\delta^{2(1-s)}\left(\varepsilon+\delta^{2(1-s)}\right)^{\alpha}.
\end{aligned}$$

□

We can finally prove our main result.

*Proof of Theorem 1.1.* Let be $\varepsilon > 0$ and $\lambda > 0$. Applying Lemma 3.4 with $\delta^{2(1-s)} := \varepsilon\lambda$, we have that

$$\langle U+\phi_{\varepsilon,\delta}-\mathcal{I}_{\delta}(g_{\varepsilon,\delta}(U+\phi_{\varepsilon,\delta})),\psi\rangle=\varepsilon\left(A+\lambda B+\frac{o(\varepsilon+\varepsilon\lambda)}{\varepsilon}\right)=:\varepsilon F(\varepsilon,\lambda).$$

Since by Lemma 2.8,

$$A:=\int_{\mathbb{R}^n}\log(U)U^{p_1}\psi\,dx>0,\ \text{ and } B:=\langle U,\psi\rangle_s<0,$$

we can take $\lambda_0 := -A/B > 0$, that implies

$$F(\varepsilon,\lambda_0)=(1+\lambda_0)\frac{o(\varepsilon(1+\lambda_0))}{\varepsilon(1+\lambda_0)}=o(1)\ \text{ (with respect to } \varepsilon).$$

That is, $F(\varepsilon, \lambda_0) = 0$ for $\varepsilon$ small enough, as wanted. □

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