THE EVOLUTION EQUATION AND THE EIGENVALUE PROBLEM FOR THE LAPLACIAN ON A REGULAR TREE

LEANDRO M. DEL PEZZO, NICOLÁS FREVENZA AND JULIO D. ROSSI

Abstract. In this paper our main goal is to study the evolution problem associated with the Laplacian operator with Dirichlet boundary conditions in a regular tree. We prove existence and uniqueness of solutions when the initial condition is compatible with the boundary condition. Next, we deal with the asymptotic behavior of the solutions and we prove that they decay to zero exponentially fast. This decay rate is governed by the associated first eigenvalue that we also study here.

1. INTRODUCTION

Código de colores Frevenza. Las cosas importantes o relativamente importantes. Comentarios de redacción, comentarios chotos y demás.

Paper que tenemos que citar [2, 3, 4, 5]. Mirar también artículos de Solomyak sobre Laplaciano con pesos en árboles, y paper de Wojciechowski (Indiana). Me parece que sería útil no solo citarlos sino relacionar nuestras cosas con lo que hacen ahí (que hay que entender).

1.1. Notations and statements of the main results.

Let us first introduce some notations needed for the precise statement of the results contained in this paper.

Let $m \in \mathbb{N}_{\geq 2}$. A regular $m-$branching tree $\mathbb{T}_m$ is a graph such that

- $\mathbb{T}_m$ consists of a root (denoted by $\emptyset$) and all finite sequences $(\emptyset, a_1, a_2, \dots, a_k)$ with $k \in \mathbb{N}$, whose coordinates a_i are chosen from $\{0, 1, \dots, m-1\}$. As usual the elements in $\mathbb{T}_m$ are called vertices or nodes;
- Each vertex x has m successors, obtained by adding another coordinate.

Notice that, if $x \neq \emptyset$ then x has a only an immediate predecessor (or ancestor), which we will denote as $\hat{x}$. We say that a vertex $x \in \mathbb{T}_m$ has $k-$level ($k \in \mathbb{N}$) if $x = (\emptyset, a_1, a_2, \dots, a_k)$ and we write $|x| = k$.

An infinite sequence of vertices, each followed by an immediate successor, is called a branch of $\mathbb{T}_m$. The collection of all branches forms the boundary of $\mathbb{T}_m$, denoted by $\partial\mathbb{T}_m$. Observe that the map $\psi : \partial\mathbb{T}_m \to [0,1]$ defined as

$$\psi(\pi) := \sum_{k=1}^{+\infty} \frac{a_k}{m^k}$$

is surjective, where $\pi = (\emptyset, a_1, \dots, a_k, \dots) \in \partial\mathbb{T}_m$ and $a_k \in \{0, 1, \dots, m-1\}$ for all $k \in \mathbb{N}$. Whenever $x = (\emptyset, a_1, \dots, a_k)$ is a vertex, the map ψ is extended by setting

$$\psi(x) \coloneqq \psi(a_1, \dots, a_k, 0, \dots, 0, \dots).$$

In this context, we first focus on studying the principal eigenvalue of a certain type of $\beta-$Laplace operator, that is

$$\Delta_\beta u(x) \coloneqq \begin{cases} \frac{1}{m}\displaystyle\sum_{i=0}^{m-1} u(\emptyset, i) - u(\emptyset) & \text{if } x = \emptyset, \\ \left(\beta u(\hat{x}) + \frac{1-\beta}{m}\displaystyle\sum_{i=0}^{m-1} u(x, i) - u(x)\right) p_\beta^{-|x|} & \text{if } x \in \mathbb{T}_m \setminus \{\emptyset\}. \end{cases}$$

Here $u\colon \mathbb{T}_m \to \mathbb{R}$ is a function, $\beta \in [0, \frac{1}{2})$ and

$$p_\beta \coloneqq \begin{cases} 1 & \text{if } \beta = 0, \\ \frac{\beta}{1-\beta} & \text{if } \beta \in (0, \frac{1}{2}). \end{cases}$$

Hay que decir algo de por qué ponemos factor asociado a p_β en Δ_β, y por qué tomamos $\beta \in (0, \frac{1}{2})$.

We now introduce our notion of the eigenvalue of Δ_β.

eigen_def

Definition 1.1. We say that $\lambda \in \mathbb{R}$ is a eigenvalue of Δ_β if there is a function $u\colon \mathbb{T}_m \to \mathbb{R}$ such that $u \not\equiv 0$ and

$$\begin{cases} -\Delta_\beta u = \lambda u & \text{on } \mathbb{T}_m, \\ \displaystyle\lim_{x\to\pi} u(x) = 0 & \text{for } \pi \in \partial\mathbb{T}_m. \end{cases}$$

In which case, we say that u is an eigenfunction associated to λ.

In this context, we have the following notion of principal eigenvalue.

Definition 1.2. A positive eigenvalue λ for Δ_β is called a principal eigenvalue if there is a non-negative eigenfunction u associated to λ.

Caso $\beta = 0$ fue tratado en [5]. Comentar resultado y decir que nos vamos a concentrar en el caso $\beta \in (0, 1/2)$

We are now in a position to formulate our results relative to the principal eigenvalue.

Revisar al final porque nos falta ver que existe la autofunción y hay que ver qué implicaciones están probadas y cuáles no.

mer_teorema

Theorem 1.3. *Let $\beta \in (0, \frac{1}{2})$. The principal eigenvalue of Δ_β is*

$$\lambda_1(\beta) = \sup \mathcal{A}_\beta$$

where

$$\mathcal{A}_\beta \coloneqq \{\lambda > 0\colon \exists v\colon \mathbb{T}_m \to \mathbb{R} \text{ with } 0 < c < v < K, \text{ satisfying } \lambda v + \Delta_\beta v \le 0 \text{ in } \mathbb{T}_m\}.$$

Moreover

$$\frac{(1-2\beta)^2}{\beta^2+(1-\beta)^2} \leq \lambda_1(\beta) \leq \frac{2(1-\beta)^2}{1+\sqrt{1-4\beta(1-\beta)^2}} < 1.$$

Finally, there is a positive eigenfunction $v\colon \mathbb{T}_m \to \mathbb{R}$ *associated to* $\lambda_1(\beta)$ *such that* v *is constant at each level,* i.e., $v(x) = f(|x|)$ *where* $f\colon \mathbb{N}_0 \to \mathbb{R}$.

Poner algo de verso de cómo es la prueba.

We begin by observing that, if $k \in \mathbb{N}$ and $u\colon \mathbb{T}_m \to \mathbb{R}$ is a function such that $u(x) = u_{|x|}$ with $u_j = 0$ for all $j > k$, the Laplacian of u could be written as

$$-\Delta_\beta u(x) = (A_k\bar{u})_{|x|}$$

for any $x \in \mathbb{T}_m$ such that $|x| \leq k$, where $\bar{u} = (u_0, \dots, u_k) \in \mathbb{R}^{k+1}$, and $A_k = (a_{i,j})$ is the matrix defined by

ef-matriz-A

$$a_{i,j} = \begin{cases} p_\beta^{-i} & \text{if } j = i, \text{ for } i = 0, \dots, k \\ -\beta p_\beta^{-i} & \text{if } j = i-1, \text{ for } i = 1, \dots, k \\ -1 & \text{if } i = 0,\ j = 1 \\ -(1-\beta)p_\beta^{-i} & \text{if } j = i+1, \text{ for } i = 1, \dots, k \end{cases}, \tag{1.1}$$

and $(A_k\bar{u})_i$ denotes the $i-$th coordinate of the vector $A_k\bar{u}$. Note that the index set for the matrix A starts from 0 to be consistent with the notation of $\bar{u}$.

Volví a la notación donde los índices de la matriz empiezan en 0. Es más conveniente porque el vector al que se le aplica también empieza en 0, porque u_0 es la ecuación a nivel 0.

Note that A_k is a $(k+1)$ square tridiagonal matrix. Using some linear algebra, we show that A_k is diagonalizable for any k, so, all eigenvalues are real. Moreover, all eigenvalues are positive. Let α_k be the smallest eigenvalue of A_k. The upper bound for λ_1 in Theorem 1.3 is exactly α_1. The following result improves this bound.

Nos gustaría probar la igualdad, pero aún (creo) que no lo tenemos.

ndo_teorema

Theorem 1.4. *The sequence* $\{\alpha_k\}_{k\geq 1} \subset \mathbb{R}^+$ *is strictly decreasing and* $\lambda_1 \leq \bar{\lambda}_1 = \inf_{k\in\mathbb{N}} \alpha_k$.

Si tenemos la igualdad tenemos el siguiente resultado para λ_1.

Remark 1.5. Note that $\bar{\lambda}_1$ does not depend on m.

Our second goal in the article is to study the the evolution equation for Δ_β. Given a initial profile $f\colon \mathbb{T}_m \to \mathbb{R}$, consider the problem to find $u\colon \mathbb{T}_m \times [0, +\infty) \to \mathbb{R}$ such that

c.evolucion

$$\begin{cases} u_t(x,t) - \Delta_\beta u(x,t) = 0 & \text{in } \mathbb{T}_m \times (0, +\infty) \\ u(x,0) = f(x) & \text{in } \mathbb{T}_m \end{cases} \tag{1.2}$$

It should be highlighted that, in the case $\beta = 0$, this evolution equation was first studied in [5].

Explicar un poco más del comportamiento en ese caso (DPMR), que por lo que recuerdo era un poco atípico, no? Por como era el decaimiento.

To enunciate an evolution theorem associated to the Laplacian Δ_β, we need to define the space where the solution lives. We write L^∞_{loc} for the space

$$L^\infty_{loc} := \{v \in L^\infty(\mathbb{T}_m \times [0,T]) \text{ for all } T \geq 0\}.$$

a_evolucion **Theorem 1.6.** *Let $\beta \in (0, \frac{1}{2})$ and $f \colon \mathbb{T}_m \to \mathbb{R}$ be a bounded function. Then, there is a unique solution $u \in L^\infty_{loc}$ for the equation* (1.1)*. Moreover*

$$|u(x,t)| \leq \|f\|_\infty e^{-\lambda_1(\beta)t}$$

where $\|f\|_\infty = \sup\limits_{x \in \mathbb{T}_m} |f(x)|$ and $\lambda_1(\beta)$ is the principal eigenvalue of Δ_β.

Sobre este capítulo habrá que volver una vez que los resultados estén probados.

Falta lo relativo a otros autovalores. Ya veremos qué tenemos probado sobre eso.

Open problem: $\lambda_1(\beta)$ es simple y aislado? La cota inferior es optima?

The paper is organized as follows. In Section 2, we show the comparison principles for the elliptic and parabolic problems. Then, in Section 3, we prove Theorems 1.3 and 1.4; Lastly, in Section 4, we show Theorem 1.6.

cp

2. COMPARISON PRINCIPLES

In this section, we prove the comparison principles for the elliptic and parabolic problems. We start by showing a maximum principle.

teo_mp **Theorem 2.1** (Maximum principle)**.** *Let $\beta \in [0, \frac{1}{2})$ and $u \colon \mathbb{T}_m \to \mathbb{R}$ be a function such that*

eq:mp1
$$-\Delta_\beta u(x) \geq 0 \quad \text{in } \mathbb{T}_m, \tag{2.3}$$

eq:mp2
$$\liminf_{x \to \pi} u(x) \geq 0 \quad \text{for } \pi \in \partial\mathbb{T}_m. \tag{2.4}$$

Then

$$\inf_{x \in \mathbb{T}_m} u(x) \geq 0.$$

Moreover, if $\beta > 0$ then either $u(x) > 0$ in $\mathbb{T}_m$ or $u(x) \equiv 0$ in $\mathbb{T}_m$.

Proof. The case $\beta = 0$ is a consequence of [4, Lemma 3.3]. For this reason we assume that $\beta > 0$.

First, we show that u is bounded from below. By contradiction, suppose that there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{T}_m$ such that $u(x_n) \to -\infty$ as $n \to \infty$. Since $\{\psi(x_n)\}_{n\in\mathbb{N}} \subset [0,1]$, there exist t_0 and a subsequence, which for simplicity we still denote by $\{\psi(x_n)\}_{n\in\mathbb{N}}$, such that

$\psi(x_n) \to t_0$ as $n \to \infty$. Therefore, we can assume that $x_n \to \pi$ for some $\pi \in \partial\mathbb{T}_m$. But then, using (2.1), we can derive the contradiction

$$0 \leq \liminf_{n\to\infty} u(x_n) = -\infty.$$

Then,

$$m_0 := \inf_{x\in\mathbb{T}_m} u(x) > -\infty.$$

Suppose that, there is $x_0 \in \mathbb{T}_m$ such that $m_0 = u(x_0)$. Then, using (2.1), we have that

$$0 \leq -\Delta_\beta u(x_0) \leq 0,$$

that is $-\Delta_\beta u(x_0) = 0$. This implies that $u(\hat{x}) = u((x_0, 1)) = \cdots = u((x_0, m-1)) = u(x_0) = m_0$. By repeating this procedure, we can show that $u(x) \equiv m_0$ in $\mathbb{T}_m$. Thus, then either $u(x) > 0$ in $\mathbb{T}_m$ or $u(x) \equiv 0$ in $\mathbb{T}_m$.

On the other hand, if m_0 is not attained in $\mathbb{T}_m$, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $u(x_n) \to m_0$ as $n \to \infty$. As before, since $\{\psi(x_n)\}_{n\in\mathbb{N}} \subset [0, 1]$, upon passing to a subsequence if necessary, we can assume that $x_n \to \pi \in \partial\mathbb{T}_m$. Thus by (2.1), we get $m_0 \geq 0$ and therefore $u(x) > 0$ in $\mathbb{T}_m$. □

A relevant consequence of the maximum principle is the following comparison principle. See also [4, 2] for others comparison principles in the context of trees for similar Laplacians. Para que no quede mal deberíamos citar a alguien más que a nosotros.

`teo_cp_e` **Theorem 2.2** (Comparison principle)**.** *Let $\beta \in [0, \frac{1}{2})$ and $u, v\colon \mathbb{T}_m \to \mathbb{R}$ be bounded function such that*

`eq:cp1`
`eq:cp2`

$$\begin{aligned} -\Delta_\beta u(x) &\geq -\Delta_\beta v(x) \qquad \textit{in } \mathbb{T}_m,\\ \liminf_{x\to\pi} u(x) &\geq \liminf_{x\to\pi} v(x) \qquad \textit{for } \pi \in \partial\mathbb{T}_m. \end{aligned}$$

Then

$$u(x) \geq v(x) \quad \textit{in } \mathbb{T}_m.$$

Moreover, if $\beta > 0$ then either $u(x) > v(x)$ in $\mathbb{T}_m$ or $u(x) \equiv v(x)$ in $\mathbb{T}_m$.

Now we give the comparison principle for the parabolic problem. The proof is analogous to the case with $\beta = 0$ (see [5, Theorem 1.2]) and we repeat it for completeness.

It is easy to check that u is a solution of the evolution equation (1.1)

Our second goal in the article is to study the the evolution equation for Δ_β, that is

$$\begin{cases} u_t(x,t) - \Delta_\beta u(x,t) = 0 & \text{in } \mathbb{T}_m \times (0, +\infty) \\ u(x, 0) = f(x) & \text{in } \mathbb{T}_m \end{cases}$$

Note that $u\colon \mathbb{T}_m \times [0, +\infty) \to \mathbb{R}$ is a solution of the evolution equation if only if

$$u(x,t) = K_f^\beta u(x,t)$$

where

`Kbeta`
$$K_f^{\beta} u(x,t) := e^{-tp_\beta^{-|x|}} f(x) + \int_0^t e^{(s-t)p_\beta^{-|x|}} k_\beta u(x,s)\, ds \tag{2.5}$$

with

$$k_\beta u(x,t) := \begin{cases} \frac{1}{m} \sum\limits_{i=0}^{m-1} u((\emptyset,i),t) & \text{if } x = \emptyset, \\ \left(\beta u(\hat{x},t) + \frac{1-\beta}{m} \sum\limits_{i=0}^{m-1} u((x,i),t) \right) p_\beta^{-|x|} & \text{if } x \neq \emptyset. \end{cases}$$

`teo_cp_p` **Theorem 2.3** (Parabolic comparison principle)**.** *Let $\beta \in [0, \frac{1}{2})$, and $f, g \in L^\infty(\mathbb{T}_m)$. If*

$$f(x) \geq g(x) \ \text{ in } \mathbb{T}_m$$

and $u, v \in L^\infty_{loc}$ are such that

$$u(x,t) \geq K_f^{\beta} u(x,t) \quad \text{ and } \quad v(x,t) \leq K_g^{\beta} v(x,t) \quad \text{ in } \mathbb{T}_m \times [0,\infty).$$

Then

$$u(x,t) \geq v(x,t) \quad \text{ in } \mathbb{T}_m \times [0,\infty).$$

Proof. Poner prueba de teorema 1.2 de DPMR3 porque creo que hay que adaptarla un poco. Cualquier cosa después lo arreglamos.

□

`pe`

3. PRINCIPAL EIGENVALUE

In this section we study the principal eigenvalue of Δ_β. We begin by showing that all eigenfunctions are bounded and that all eigenvalues are positive. We assume that $\beta \in (0, \frac{1}{2})$.

`lema_fe_1` **Lemma 3.1.** *Let $\beta \in (0, \frac{1}{2})$, λ be an eigenvalue of Δ_β and u be an eigenfunction associated to λ. Then $\lambda > 0$ and u is bounded.*

Proof. We start by showing that u is bounded. On the contrary, if u is not bounded there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{T}_m$ such that $|u(x_n)| \to \infty$ as $n \to \infty$. Without loss of generality, we may assume that $u(x_n) \to \infty$. On the other hand, since $\{\psi(x_n)\}_{n\in\mathbb{N}} \subset [0,1]$, upon passing to a subsequence if necessary, we can assume that $x_n \to \pi \in \partial\mathbb{T}_m$. Then, by the boundary condition of the eigenvalue problem 1.1, we have

$$0 = \lim_{n\to\infty} u(x_n) = \infty,$$

and this contradiction shows that u is bounded.

We now prove that $\lambda > 0$. We again proceed by contradiction. We assume that $\lambda \leq 0$.

If $\lambda = 0$ then $u \equiv 0$ in $\mathbb{T}_m$ by Theorem 2.2. Since u is an eigenfunction we get a contradiction.

Now we study the case $\lambda < 0$. We claim that any eigenfunction u satisfies that $u \leq 0$ in $\mathbb{T}_m$. Assuming this claim, since $-u$ is also an eigenfunction associated to λ, we have that $-u(x) \leq 0$ in $\mathbb{T}_m$. Thus $u \equiv 0$, that is a contradiction.

To finish the proof we need to show the claim. Once again, we proceed by contradiction. Suppose that

$$M := \sup_{x \in \mathbb{T}_m} u(x) > 0.$$

Using the boundary condition, it is easy to check that the supremum is a maximum and there is $x_0 \in \mathbb{T}_m$ with $u(x_0) = M$. Then, since u is an eigenfunction for λ and we are assuming that $\lambda < 0$, we get

$$0 \leq -\Delta_\beta u(x_0) = \lambda u(x_0) \leq 0.$$

Then $u((x_0, i)) = M$ for any $i = 0, \dots, m-1$ and $u(\hat{x}_0) = M$. By repeating this procedure, it follows that $u(x) = M$ for any $x \in \mathbb{T}_m$, and it contradicts the boundary condition. So, the claim is proved and this concludes the proof. $\square$

cte_niveles

Lemma 3.2. *Let λ be an eigenvalue of Δ_β and u be an eigenfunction associated to λ. Then, $\bar{u} \colon \mathbb{T}_m \to \mathbb{R}$ defined by*

$$\bar{u}(x) = \frac{1}{m^{|x|}} \sum_{y \in \mathbb{T}_m \colon |y| = |x|} u(y)$$

is also an eigenfunction associated to λ, which is constant at each level of the tree.

Proof. Note that $\bar{u}$ depends only on $|x|$. Using this fact it is easy to check that $\Delta_\beta \bar{u}(x)$ can be written as

$$\Delta_\beta \bar{u}(x) = \left(\beta \bar{u}(\hat{x}) + (1-\beta)\bar{u}(x,0) - \bar{u}(x)\right) p_\beta^{-|x|} \qquad \text{if } x \neq \emptyset.$$

Now, appliying the definition of $\bar{u}$ we obtain

$$\Delta_\beta \bar{u}(x) = \left(\beta \frac{1}{m^{|\hat{x}|}} \sum_{\substack{y \in \mathbb{T}_m \\ |y| = |\hat{x}|}} u(y) + (1-\beta) \frac{1}{m^{|(x,0)|}} \sum_{\substack{y \in \mathbb{T}_m \\ |y| = |(x,0)|}} u(y) - \frac{1}{m^{|x|}} \sum_{\substack{y \in \mathbb{T}_m \\ |y| = |x|}} u(y) \right) p_\beta^{-|x|}.$$

Re-writting the sums as

$$\sum_{\substack{y \in \mathbb{T}_m \\ |y| = |(x,0)|}} u(y) = \sum_{\substack{y \in \mathbb{T}_m \\ |y| = |x|}} \sum_{i=0}^{m-1} u(y,i) \qquad \text{and} \qquad \sum_{\substack{y \in \mathbb{T}_m \\ |y| = |\hat{x}|}} u(y) = \frac{1}{m} \sum_{\substack{y \in \mathbb{T}_m \\ |y| = |x|}} u(\hat{y}),$$

we have for $x \neq \emptyset$

$$\begin{aligned}\Delta_\beta \bar{u}(x) &= \frac{1}{m^{|x|}} \left(\beta \sum_{\substack{y\in\mathbb{T}_m\\|y|=|x|}} u(\hat{y}) + \frac{1-\beta}{m} \sum_{\substack{y\in\mathbb{T}_m\\|y|=|x|}} \sum_{i=0}^{m-1} u(y,i) - \sum_{\substack{y\in\mathbb{T}_m\\|y|=|x|}} u(y) \right) p_\beta^{-|x|} \\ &= \frac{1}{m^{|x|}} \sum_{\substack{y\in\mathbb{T}_m\\|y|=|x|}} \left(\beta u(\hat{y}) + \frac{1-\beta}{m} \sum_{i=0}^{m-1} u(y,i) - u(y) \right) p_\beta^{-|y|} \\ &= \frac{1}{m^{|x|}} \sum_{\substack{y\in\mathbb{T}_m\\|y|=|x|}} \Delta_\beta u(y).\end{aligned}$$

So, when $x \neq \emptyset$ it follows that

$$\lambda \bar{u}(x) + \Delta_\beta \bar{u}(x) = \frac{1}{m^{|x|}} \sum_{\substack{y\in\mathbb{T}_m\\|y|=|x|}} [\lambda u(y) + \Delta_\beta u(y)] = 0,$$

since u is an eigenfunction related to λ.

Finally, note that in the case $x = \emptyset$, we have $\bar{u}(\emptyset) = u(\emptyset)$, and since $\bar{u}$ is constant at each level, we write

$$\lambda \bar{u}(\emptyset) + \Delta_\beta \bar{u}(\emptyset) = \lambda u(\emptyset) + \bar{u}(\emptyset, 0) - u(\emptyset) = \lambda u(\emptyset) + \sum_{\substack{y\in\mathbb{T}_m\\|y|=1}} u(y) - u(\emptyset) = 0.$$

Hence, $\bar{u}$ is an eigenfuction associated to λ. □

Our first result about to the principal eigenvalue shows that it is less than 1.

`lema_fe_2` **Lemma 3.3.** *Let $\beta \in (0, \frac{1}{2})$, and λ be a principal eigenvalue of Δ_β. Then $\lambda < 1$.*

Proof. Let u be the non-negative eigenfunction associated to λ. By Theorem 2.1, we have that $u(x) > 0$ in $\mathbb{T}_m$. On the other hand, since

$$\lambda u(\emptyset) = u(\emptyset) - \frac{1}{m} \sum_{i=0}^{m-1} u(\emptyset, i),$$

we have that

$$(1-\lambda) u(\emptyset) = \frac{1}{m} \sum_{i=0}^{m-1} u(\emptyset, i) > 0.$$

Therefore $\lambda < 1$. □

Lemma 3.4. *Let λ and θ be principal eigenvalues of the operator Δ_β such that $\lambda < \theta$. Call u and v for their eigenfunctions respectively. Assume that u and v are constant at each level of $\mathbb{T}_m$ and coincides at the root of $\mathbb{T}_m$, that is, $u(x) = a_{|x|}$ and $v(x) = b_{|x|}$ for all x, where $\{a_k\}$ and $\{b_k\}$ are positive sequences with $a_0 = b_0$. Then, $u \geq v$, that is, $a_k \geq b_k$ for all k.*

Proof. Normalizing, since both sequences are positive, we assume that $a_0 = b_0 = 1$. Using that u and v are eigenvalues at level 0, we have that $a_1 > b_1$ since

$$a_1 = 1 - \lambda \qquad \text{and} \qquad b_1 = 1 - \theta.$$

We proceed by induction in k. So, we assume that $a_l > b_l$ for each $l \leq k$. By the equation $\lambda u(x) + \Delta_\beta u(x) = 0$ with $|x| = k$, we have that

$$a_{k+1} = \frac{(1 - \lambda p_\beta^k) a_k - \beta a_{k-1}}{1 - \beta}.$$

Define an auxiliar function $f \colon X \to \mathbb{R}$ by the formula $f(x, y, z) = \frac{(1-z)x - \beta y}{1 - \beta}$ where the set X is

$$X = \{(x, y, z) \in \mathbb{R}^3 \colon x > 0,\ y > 0,\ 0 < z < 1, \}.$$

Note that all partial derivatives of f are negative. Then, using the induction hypothesis and the fact that $\lambda < \theta$, we have

$$\begin{aligned} a_{k+1} = f(a_k, a_{k-1}, \lambda p_\beta^k) > f(b_k, b_{k-1}, \lambda p_\beta^{k+1}) &= \frac{(1 - \lambda p_\beta^{k+1}) b_k - \beta b_{k-1}}{1 - \beta} \\ &> \frac{(1 - \theta p_\beta^{k+1}) b_k - \beta b_{k-1}}{1 - \beta} = b_{k+1}. \end{aligned}$$

So, the result follows. □

We define $\lambda_1(\beta)$ as

`ambda_1:def`

$$\lambda_1(\beta) \coloneqq \sup \mathcal{A}_\beta \tag{3.6}$$

where

`conjunto-A`

$$\mathcal{A}_\beta = \{\lambda > 0 \colon \exists v \colon \mathbb{T}_m \to \mathbb{R} \text{ with } 0 < c \leq v < K, \text{ satisfying } \lambda v + \Delta_\beta v \leq 0 \text{ in } \mathbb{T}_m\}.$$

Motivated by [1], we want to prove that $\lambda_1(\beta)$ is the unique principal eigenvalue of Δ_β.

Cuidado porque aún no tenemos esto.

First, we show that $\lambda_1(\beta)$ is well-defined and positive.

`:A_no_vacio` **Lemma 3.5.** *Let* $\beta \in (0, \frac{1}{2})$. *Then* $\mathcal{A}_\beta \neq \emptyset$.

Proof. Consider the function $v \colon \mathbb{T}_m \to \mathbb{R}$ defined by

$$v(x) = 1 + (|x| + a) p_\beta^{|x|}.$$

where $a > 0$ is a parameter that will be chosen later. Note that v is uniformly positive and bounded above by $K > 0$. Since v is constant at each level of $\mathbb{T}_m$, it is easy to compute $\Delta_\beta v(x)$ for any $x \in \mathbb{T}_m$. In fact, we have

$$\Delta_\beta v(\emptyset) = a(p_\beta - 1) + p_\beta, \qquad \text{and} \qquad \Delta_\beta v(x) = 2\beta - 1 \quad \forall x \in \mathbb{T}_m \setminus \{\emptyset\}.$$

Since $p_\beta < 1$, when a is large enough we have that $\Delta_\beta v(x) \le 2\beta - 1 < 0$ for all $x \in \mathbb{T}_m$. Then, taking $\lambda > 0$ small enough we have

$$\lambda v + \Delta_\beta v \le \lambda K + 2\beta - 1 < 0 \text{ in } \mathbb{T}_m,$$

and $\mathcal{A}_\beta \neq \emptyset$ follows. □

We have the following bound for λ_1.

a_1<alpha_1 **Lemma 3.6.** *Let* $\beta \in (0, \frac{1}{2})$. *Then*

$$0 < \lambda_1(\beta) \le \frac{2(1-\beta)^2}{1 + \sqrt{1 - 4\beta(1-\beta)^2}} < 1.$$

Proof. Let $\alpha_1 > 0$ be the minimum real number such that there exists a non-negative function $u\colon \mathbb{T}_m \to \mathbb{R}$, where $u(x)$ depends only on $|x|$ and u satisfies

autovalores

$$\alpha_1 u(x) + \Delta_\beta u(x) = 0 \quad \text{for } |x| \le 1 \quad \text{and} \quad u(x) = 0 \quad \text{for } |x| \ge 2. \tag{3.7}$$

Equation (3) could be written as an eigenvalue problem for a 2×2 matrix. So, it is easy to compute that

alpha_1

$$\alpha_1 = \frac{1}{2\beta}\left(1 - \sqrt{1 - 4\beta(1-\beta)^2}\right) = \frac{2(1-\beta)^2}{1 + \sqrt{1 - 4\beta(1-\beta)^2}}.$$

and

ion-alpha_1

$$u(\emptyset) = a, \qquad u(\emptyset, i) = (1 - \alpha_1)a \quad \forall i = 0, \dots, m-1.$$

Observe that $0 < \alpha_1 < 1$. Then $u \ge 0$ if $a > 0$. We have also that u increases as we increase a. We will choose later an appropriate a.

We claim that $\lambda_1(\beta) \le \alpha_1$. Suppose that $\lambda_1(\beta) > \alpha_1$. Then, there exists $\lambda > \alpha_1$ and $v\colon \mathbb{T}_m \to \mathbb{R}$ with $0 < c < v < K$ such that

$$\lambda v + \Delta_\beta v \le 0,$$

or, equivalently,

bda-alpha_1

$$\alpha_1 v + (\lambda - \alpha_1)v + \Delta_\beta v \le 0. \tag{3.8}$$

Let $a = u(\emptyset)$ be such that $u \le v$ for all $x \in \mathbb{T}_m$ but $u(x_0) = v(x_0)$ for some x_0 with $|x_0| \le 1$. Now, taking the difference between (3) and (3) combined with the choice of a, we obtain

cion-lambda

$$\alpha_1(v-u)(x_0) + (\lambda - \alpha_1)v(x_0) + \Delta_\beta(v-u)(x_0) = (\lambda - \alpha_1)v(x_0) + \Delta_\beta(v-u)(x_0) \le 0. \tag{3.9}$$

Note that the term $(\lambda - \alpha_1)v(x_0)$ is positive since $v > c > 0$ and the assumption $\lambda > \alpha_1$. Using that $u(x_0) = v(x_0)$ we have

$$\Delta_\beta(v-u)(x_0) = \tfrac{1}{m}\sum_i (v-u)(\emptyset, i) \qquad \text{if } x_0 = \emptyset,$$

or

$$\Delta_\beta(v-u)(x_0) = \left(\beta(v-u)(\emptyset) + \tfrac{1-\beta}{m}\sum_i v(\emptyset,j,i))\right) p_\beta^{-1} \qquad \text{if } x_0 = (\emptyset, j).$$

In both cases $\Delta_\beta(v-u)(x_0) \geq 0$ due to the fact that $v \geq u \geq 0$. So, it contradicts (3) and $\lambda_1(\beta) \leq \alpha_1$. ☐

Lemma 3.7. *Let $\lambda > 0$ be a principal value and consider $\theta \in \mathcal{A}_\beta$. Then, $\lambda \geq \theta$.*

Proof. Suppose that $\theta > \lambda$. Let u be the positive eigenfunction associated to λ,. By Lemma 3.1, we have that u is bounded. Consider also v be the function such that $\theta v(x) + \Delta_\beta v(x) \leq 0$ and $0 < c < v(x) < K$ for all $x \in \mathbb{T}_m$.

For all $x \in \mathbb{T}_m$ have that

$$-v(x)\Delta_\beta u(x) = \lambda u(x)v(x) \qquad \text{and} \qquad -u(x)\Delta_\beta v(x) \geq \theta u(x)v(x).$$

Then,

$$\begin{aligned} 0 < (\theta-\lambda)u(\emptyset)v(\emptyset) &\leq v(\emptyset)\Delta_\beta u(\emptyset) - u(\emptyset)\Delta_\beta v(\emptyset) \\ &= \frac{1}{m}\sum_{i=0}^{m} \big(u(\emptyset,i)v(\emptyset) - v(\emptyset,i)u(\emptyset)\big) \\ &= \frac{1}{m}\sum_{i=0}^{m} \left(\frac{u(\emptyset,i)}{v(\emptyset,i)} - \frac{u(\emptyset)}{v(\emptyset)}\right) v(\emptyset)v(\emptyset,i), \end{aligned}$$

and for any $x \in \mathbb{T}_m \setminus \{\emptyset\}$ we get

$$\begin{aligned} &0 < (\theta-\lambda)u(x)v(x)p_\beta^{|x|} \\ &\leq \beta\left(u(\hat{x})v(x) - v(\hat{x})u(x)\right) + \frac{1-\beta}{m}\sum_{i=0}^{m}\left(u(x,i)v(x) - v(x,i)u(x)\right) \\ &\leq \beta\left(\frac{u(\hat{x})}{v(\hat{x})} - \frac{u(x)}{v(x)}\right)v(\hat{x})v(x) + \frac{1-\beta}{m}\sum_{i=0}^{m}\left(\frac{u(x,i)}{v(x,i)} - \frac{u(x)}{v(x)}\right)v(x)v(x,i). \end{aligned}$$

It is easy to check that the function $\frac{u}{v}$ achieves a maximum at some point $x_0 \in \mathbb{T}_m$, since is positive, v is bounded away from 0, and u goes to 0 at the boundary. Therefore, if $x_0 = \emptyset$,

$$0 < \frac{1}{m}\sum_{i=0}^{m}\left(\frac{u(\emptyset,i)}{v(\emptyset,i)} - \frac{u(\emptyset)}{v(\emptyset)}\right)v(\emptyset)v(\emptyset,i) \leq 0$$

or if $x_0 \neq \emptyset$,

$$0 < \beta\left(\frac{u(\hat{x}_0)}{v(\hat{x}_0)} - \frac{u(x_0)}{v(x_0)}\right)v(\hat{x}_0)v(x_0) + \frac{1-\beta}{m}\sum_{i=0}^{m}\left(\frac{u(x_0,i)}{v(x_0,i)} - \frac{u(x_0)}{v(x_0)}\right)v(x_0)v(x_0,i) \leq 0,$$

we have a contradiction. Thus $\lambda \geq \theta$. ☐

An immediate consequence of the previous lemma is the following corollary.

Corollary 3.8. *Let $\lambda > 0$ be a principal value. Then, $\lambda \geq \lambda_1$, where λ_1 is defined in* (3).

Es casi lo mismo que lo anterior.

Lemma 3.9. *Let $\lambda > 0$ be an eigenvalue associated to a non-negative eigenfunction $u\colon \mathbb{T}_m \to \mathbb{R}$. Then $\lambda \geq \lambda_1$.*

Proof. We proceed by contradiction using the Parabolic comparison principle 2.3. Suposse that $\lambda < \lambda_1$. Then, there exists $\theta \in \mathcal{A}_\beta$ and $v\colon \mathbb{T}_m \to \mathbb{R}$ with $0 < c \leq v \leq K$ such that

le_9_ec_aux

$$\theta v(x) + \Delta_\beta v(x) \leq 0, \tag{3.10}$$

and $\theta > \lambda$.

Since the eigenfunction u is bounded by Lemma 3.1, we can choose $L > 0$ large enough such that $u(x) \leq Lv(x)$.

Define the auxiliar functions $U, V\colon \mathbb{T}_m \times [0, +\infty) \to \mathbb{R}$ by

$$U(x,t) = e^{-\lambda t} u(x) \qquad \text{and} \qquad V(x,t) = Le^{-\theta t} v(x).$$

Note that $U(x,0) = u(x) \leq Lv(x) = V(x,0)$ and the fact that both functions are non-negative. Assume that for all $x \in \mathbb{T}_m$ and $t \in [0, +\infty)$ we have

e_9_ec_aux2

$$K^{\beta}_{u} U(x,t) \geq U(x,t) \qquad \text{and} \qquad K^{\beta}_{Lv} V(x,t) \leq V(x,t), \tag{3.11}$$

where $K^{\beta}_{\cdot}$ is the operator defined in (2) and the sub-index denotes the initial condition. Then, the Parabolic comparison principle implies that

$$V(x,t) \geq U(x,t) \geq 0 \qquad \text{in } \mathbb{T}_m \times [0, +\infty),$$

which is a contradiction, since for fixed x, the function $V(x,t)$ tends to zero faster than $U(x,t)$ as $t \to \infty$ due to $\theta > \lambda$.

We need to prove the claim (3) for complete the Lemma. By (3), when $x \neq \emptyset$ holds

$$\left(\beta v(\hat{x}) + \frac{1-\beta}{m} \sum_{i=0}^{m-1} v(x,i) \right) p_\beta^{-|x|} \leq -\theta v(x) + p_\beta^{-|x|} v(x).$$

Then, using the previous inequality and the definition of V we have for $x \neq \emptyset$

$$\begin{aligned}
K^{\beta}_{Lv} V(x,t) &= Lv(x) e^{-tp_\beta^{-|x|}} + \int_0^t e^{(s-t)p_\beta^{|x|}} \left(\beta V(\hat{x}, s) + \frac{1-\beta}{m} \sum_{i=0}^{m-1} V((x,i),s) \right) p_\beta^{-|x|} \mathrm{d}s \\
&\leq Lv(x) e^{-tp_\beta^{-|x|}} + L \int_0^t e^{(s-t)p_\beta^{|x|}} \left(-\theta v(x) + p_\beta^{-|x|} v(x) \right) e^{-\theta s} \mathrm{d}s \\
&= Lv(x) e^{-tp_\beta^{-|x|}} + Le^{-tp_\beta^{|x|}} \left(-\theta v(x) + p_\beta^{-|x|} v(x) \right) \int_0^t e^{s(p_\beta^{|x|} - \theta)} \mathrm{d}s \\
&= Lv(x) e^{-t\theta} = V(x,t).
\end{aligned}$$

In the same way one proves the claim for V at $x = \emptyset$ and then for $U(x,t)$. The application of the Parabolic comparison principle follows. □

lema_suc1

Lemma 3.10. *If $\lambda \in \mathcal{A}_\beta$ then there is a function $u\colon \mathbb{T}_m \to \mathbb{R}$ such that $u(x) = a_{|x|}$ for all $x \in \mathbb{T}_m$, $0 < c \le a_k < K$, and*

$$\lambda u(x) + \Delta_\beta u(x) \le 0 \quad \forall x \in \mathbb{T}_m.$$

Moreover $\{a_k\}$ is a decreacing sequence.

Proof. Since $\lambda \in \mathcal{A}_\beta$ there exists a function $v\colon \mathbb{T}_m \to \mathbb{R}$ such that $0 < c < v < K$ in $\mathbb{T}_m$ and

$$\lambda v + \Delta_\beta v \le 0.$$

Let $\bar{v}(x)$ be the function defined by

$$\bar{v}(x) = \frac{1}{m^{|x|}} \sum_{y \in \mathbb{T}_m \colon |y| = |x|} v(y).$$

This function is constant at each level and we write $\bar{v}(x) = a_{|x|}$. It is easy to check that $0 < c < \bar{v} < K$ and arguing as in Lemma 3.2 we have that

eq:auxiliar

$$\lambda \bar{v} + \Delta_\beta \bar{v} \le 0. \tag{3.12}$$

Lastly, we show that $\{a_k\}$ is a decreasing sequence. The proof is by induction on k. For $k = 1$, by (3) we have that

$$a_1 \le (1 - \lambda) a_0 < a_0$$

since $\lambda \le \lambda_1 < 1$. Suppose now that $a_k < a_{k-1}$ and let us show that $a_{k+1} < a_k$. Thus, by (3) and our induction hypothesis, we have that

q:auxiliar2

$$a_{k+1} \le \frac{(1 - p_\beta^k \lambda) a_k - \beta a_{k-1}}{1 - \beta} < \frac{1 - p_\beta^k \lambda - \beta}{1 - \beta} a_k = \left(1 - \frac{\lambda}{1 - \beta} p_\beta^k\right) a_k. \tag{3.13}$$

Using that $\lambda \le \lambda_1$ and the upper bound of Lemma 3.6 we get

$$0 < \frac{\lambda}{1 - \beta} p_\beta^k < \frac{\lambda}{1 - \beta} p_\beta < \frac{2\beta}{1 + \sqrt{1 - 4\beta(1 - \beta)^2}} < \frac{1}{1 + \sqrt{1 - 4\beta(1 - \beta)^2}} < 1.$$

It follows from (3) that $a_{k+1} < a_k$. Therefore $\{a_k\}$ is a decreasing sequence. ☐

Now we are in condition to show that there is a positive eigenfunction associated to $\lambda_1(\beta)$.

Este lema tiene un error que no supe resolver.

_importante

Lemma 3.11. *Let $\beta \in (0, \frac{1}{2})$. Then, $\lambda_1(\beta)$ defined in (3) is a principal eigenvalue of Δ_β. Moreover there is a positive eigenfunction ω associated to $\lambda_1(\beta)$ such that $\omega(x) = a_{|x|}$ for all $x \in \mathbb{T}_m$ with $a_0 = 1$, and $a_k \ge a_{k+1} > 0$.*

Proof. For $L > 0$ define the set

$$\begin{aligned} \mathcal{S}_{\beta,L} :=\Big\{ \{a_k\} \subset \mathbb{R} \colon a_1 = L,\ a_k \ge a_{k+1} > 0,\ \lambda_1 a_0 + a_1 - a_0 \le 0, \\ \lambda_1 a_k + \left[\beta a_{k-1} + (1 - \beta) a_{k+1} - a_k\right] p_\beta^{-k} \le 0 \text{ and } \lim_{k \to \infty} a_k = 0 \Big\}. \end{aligned}$$

Observe that $\inf \mathcal{S}_{\beta,1} = \frac{1}{L} \inf \mathcal{S}_{\beta,L}$.

Lo anterior no sirve para mucho creo (en algún momento creímos que sí pero la cuenta se caga igual como explico más adelante).

We will show that the positive eigenfunction associated to $\lambda_1(\beta)$ is

genfunction

$$w(x) \coloneqq \inf\{a_{|x|} \colon \{a_k\} \in \mathcal{S}_{\beta,1}\}. \tag{3.14}$$

The proof split in two steps, we first will show that $\mathcal{S}_{\beta,1} \neq \emptyset$ and therefore w is well-defined. Then, we will prove that w is an eigenfunction associated to λ_1.

Step 1. We prove that $\mathcal{S}_{\beta,1} \neq \emptyset$.

Let $\{\mu_n\} \subset \mathcal{A}_\beta$ such that $\mu_{n+1} \geq \mu_n$ and $\mu_n \to \lambda_1$ as $n \to \infty$. By Lemma 3.10, for any $k \in \mathbb{N}$, there is a positive decreasing sequence $\{a_k^n\}$ such that

$$\mu_n a_0^n + a_1^n - a_0^n \leq 0 \text{ and } \mu_n a_k^n + \left[\beta a_{k-1}^n + (1-\beta) a_{k+1}^n - a_k^n\right] p_\beta^{-k} \leq 0 \quad \forall k \in \mathbb{N}.$$

Without loss of generality, we can assume that $a_0^n = 1$ for any $n \in \mathbb{N}$. Then we set

$$b_0 = 1 \text{ and } b_k = \liminf_{n\to\infty} a_k^n \quad \forall k \in \mathbb{N}.$$

Our aim is to show that $\{b_k\} \in \mathcal{S}_{\beta,1}$.

It is easy to check that, $\{b_k\}$ is a non-negative and non-increasing sequence such that

$$\lambda_1 b_0 + b_1 - b_0 \leq 0 \text{ and } \lambda_1 b_k + \left[\beta b_{k-1} + (1-\beta) b_{k+1} - b_k\right] p_\beta^{-k} \leq 0 \quad \forall k \in \mathbb{N},$$

that is, the function $u\colon \mathbb{T}_m \to \mathbb{R}$, $u(x) = b_{|x|}$ satisfies

$$\lambda_1 u + \Delta_\beta u \leq 0 \text{ in } \mathbb{T}_m.$$

Notice that, as $u(\emptyset) = 1$, we have $u(x) > 0$ in $\mathbb{T}_m$ owing to the maximum principle (see Theorem 2.1). Therefore $\{b_k\}$ is positive.

To conclude this first step, we need to show that $\lim\limits_{k\to\infty} b_k = 0$. To prove it, we argue by contradiction. So, let assume that

$$\lim_{k\to\infty} b_k = c > 0.$$

Then, we define a function v as

$$v(x) = u(x) - \tfrac{c}{2}.$$

Since $\{b_k\}$ is decreasing, $\frac{c}{2} \leq v(x) \leq 1 - \frac{c}{2}$ in $\mathbb{T}_m$. It follows that for all $x \in \mathbb{T}_m$

$$(\lambda_1 + \varepsilon) v(x) + \Delta_\beta v(x) = \varepsilon u(x) - (\lambda_1 + \varepsilon)\tfrac{c}{2} \leq \varepsilon \left(1 - \tfrac{c}{2}\right) - \lambda_1 \tfrac{c}{2} \leq 0$$

for any $0 < \varepsilon \leq \lambda_1 \frac{c}{2}(1 - \frac{c}{2})^{-1}$.

So, we proved that $\lambda_1 + \varepsilon \in \mathcal{A}_\beta$ for ε small enough, which contradicts the definition of λ_1 as the supremum of $\mathcal{A}_\beta$. Hence, $\lim\limits_{k\to\infty} b_k = 0$ and $\mathcal{S}_{\beta,1} \neq \emptyset$.

Step 2. We prove that there is an eigenfunction associated to $\lambda_1(\beta)$.

The function w defined in (3) depends only on the level, so, we write $w(x) = w_{|x|}$.

It easy to check that $\{w_k\}$ is non-negative, $w_0 = 1$, $w_k \geq w_{k+1}$ for any $k \in \mathbb{N}_0$ and

eq_w

$$\begin{aligned} &\lambda_1 + w_1 - 1 \leq 0, \\ &\lambda_1 w_k + [\beta w_{k-1} + (1-\beta) w_{k+1} - w_k] p_\beta^{-k} \leq 0 \text{ if } k \in \mathbb{N}_0, \\ &\lim_{k\to\infty} w_k = 0. \end{aligned} \tag{3.15}$$

Moreover, by the maximum principle, we have that w is positive.

We claim that for any $k \in \mathbb{N}$ we have

$$w_k > w_{k+1}.$$

Suppose that there is $k_0 \in \mathbb{N}_0$ such that

$$w_{k_0} = w_{k_0+1}.$$

Then by (3) we have that

$$\begin{aligned} &\lambda_1 \leq 0 \text{ if } k_0 = 0 \\ &\lambda_1 w_{k_0} + \beta(w_{k_0-1} - w_{k_0+1}) p_\beta^{-k_0} \leq 0 \text{ if } k_0 \neq 0. \end{aligned}$$

In both case we have a contradiction, because $\lambda_1 > 0$ and $w_{k_0-1} \geq w_{k_0} \geq w_{k_0+1} > 0$ for any $k \in \mathbb{N}_0$. Thus the claim holds.

To conclude the proof we need to show that

eq_w_2

$$\begin{aligned} &\lambda_1 + w_1 - 1 = 0, \\ &\lambda_1 w_k + [\beta w_{k-1} + (1-\beta) w_{k+1} - w_k] p_\beta^{-k} = 0 \text{ if } k \in \mathbb{N}, \end{aligned} \tag{3.16}$$

We argue again by contradiction. So, let assume that (3) does not hold and let k_0 be the minimum level for which one of the equations fails. In fact, we have

$$\begin{aligned} &\lambda_1 + w_1 - 1 < 0 \text{ if } k_0 = 0; \\ &\lambda_1 w_{k_0} + [\beta w_{k_0-1} + (1-\beta) w_{k_0+1} - w_{k_0}] p_\beta^{-k_0} < 0 \text{ if } k_0 \neq 0. \end{aligned}$$

Si pasa lo de la abajo, la función auxiliar de abajo lo resuelve. Si pasa lo de arriba, es decir, $k_0 = 0$

$$\lambda_1 + w_1 - 1 < 0.$$

la función auxiliar no sirve porque $\tilde{w}_0 \neq 1$ y entonces $\tilde{w} \notin \mathcal{S}_{\beta,1}$. Fui probando con otras perturbaciones de w pero siempre se me caga en algún lado. Por ejemplo, si se considera $\tilde{w}_k = w - \varepsilon p^k$, las desigualdades del laplaciano se cumplen pero no sé ni que la sucesión sea positiva ni decreciente.

We take a new function $\tilde{w}\colon \mathbb{T}_m \to \mathbb{R}$ defined by

$$\tilde{w}(x) = \tilde{w}_{|x|} := \begin{cases} w_k & \text{if } |x| = k \neq k_0, \\ w_{k_0} - \varepsilon & \text{if } |x| = k = k_0, \end{cases}$$

where $\varepsilon > 0$ will be chosen later.

It is easy to check that if ε is small enough $\{\tilde{w}_k\}$ is a non-negative decreasing sequence and $\lim_{k\to\infty} \tilde{w}_k = 0$.

Since $\tilde{w}$ changes the value of w only at k_0, we need to show that $\lambda_1\tilde{w} + \Delta_\beta\tilde{w} \leq 0$ holds at level k_0 and $k_0 \pm 1$.

If $k_0 \geq 2$, we compute

`ec_pertur`

$$\begin{aligned}
&\lambda_1\tilde{w}_k + \Delta_\beta\tilde{w}_k = \lambda_1 w_k + \Delta_\beta w_k - \varepsilon\beta && \text{if } k = k_0 + 1\\
&\lambda_1\tilde{w}_k + \Delta_\beta\tilde{w}_k = \lambda_1 w_k + \Delta_\beta w_k - \varepsilon(1-\beta) && \text{if } k = k_0 - 1\\
&\lambda_1\tilde{w}_k + \Delta_\beta\tilde{w}_k = \lambda_1 w_k + \Delta_\beta w_k + \varepsilon(p_\beta^k - \lambda_1) && \text{if } k = k_0.
\end{aligned}$$

Using that $\lambda_1 w_k + \Delta_\beta w_k \leq 0$ we show that the first and second lines are negative. For $k = k_0$ we have that $\lambda_1 w_{k_0} + \Delta_\beta w_{k_0} < 0$, so, for ε small enough the inequality remains strict. Thus, we obtain $\lambda_1\tilde{w}_k + \Delta_\beta\tilde{w}_k \leq 0$ for all k.

If $k_0 = 1$, the equation corresponding to $k = k_0 - 1 = 0$ is slightly different, but the others remains as in the previous case. In fact, at level 0 we have

$$\lambda_1\tilde{w}_0 + \Delta_\beta\tilde{w}_0 = \lambda_1 w_0 + \Delta_\beta w_0 - \varepsilon\beta - \varepsilon \leq -\varepsilon\beta - \varepsilon < 0.$$

Hence, when $k_0 \geq 1$ the sequence $\{\tilde{w}_k\} \in \mathcal{S}_{\beta,1}$ but $\tilde{w}_k \leq w_k$ and the inequality is strict at $k = k_0$, which contradicts the definition of w. Then, the system (3) is satisfied.

Acá empieza a cagarse.

In the case where $k_0 = 0$, to check that $\lambda_1\tilde{w} + \Delta_\beta\tilde{w} \leq 0$, we have only equations at level 0 and 1. In fact, we have

$$\begin{aligned}
&\lambda_1\tilde{w}_0 + \Delta_\beta\tilde{w}_0 = \lambda_1 w_0 + \Delta_\beta w_0 + \varepsilon(1 - \lambda_1) && \text{if } k = 0\\
&\lambda_1\tilde{w}_1 + \Delta_\beta\tilde{w}_1 = \lambda_1 w_1 + \Delta_\beta w_1 - \varepsilon\beta && \text{if } k = 1.
\end{aligned}$$

Then, using that $\lambda_1 w_0 + \Delta_\beta w_0 < 0$ we can choose ε small enough such that both equations are less than 0. Hence, $\tilde{w} \in \mathcal{S}_{\beta,1-\varepsilon}$ and pero cuando normalizo pierdo que esté debajo de w.

$\square$

Now, we show a lower bound for $\lambda_1(\beta)$.

Para este lema preciso que exista la autofunción.

Lemma 3.12. *Let* $\beta \in (0, \frac{1}{2})$. *Then*

$$\lambda_1(\beta) \geq \frac{(1-2\beta)^2}{\beta^2 + (1-\beta)^2}.$$

Proof. By Lemma 3.11, there is a positive eigenfunction w associated to λ_1 such that $w(x) = w_{|x|}$ for all $x \in \mathbb{T}_m$ with $w_0 = 1$, and $w_k > w_{k+1} > 0$. Then

$$\begin{aligned}
\lambda_1 &= 1 - w_1\\
\lambda_1 w_k p_\beta^k &= \beta(w_k - w_{k-1}) + (1-\beta)(w_k - w_{k+1}) \quad \forall k \geq 1.
\end{aligned}$$

Therefore, using that $\lim_{k\to\infty} w_k = 0$ we have

$$\lambda_1 \sum_{k=0}^{\infty} p_\beta^k w_k = \lambda_1 - \beta + (1-\beta) w_1 = 1 - 2\beta + \beta\lambda_1.$$

Since $w_k \le w_0 = 1$ for all k, we have

$$1 - 2\beta + \beta\lambda_1 \le \lambda_1 \sum_{k=0}^{\infty} p_\beta^k = \lambda_1 \frac{1-\beta}{1-2\beta},$$

and thus, $\lambda_1 \ge \frac{(1-\beta)^2}{\beta^2+(1-\beta)^2}$. ☐

No sé si lo que sigue es útil o no, pero se deduce fácil de una cuenta anterior. Nos dice algo sobre cómo es la velocidad con la que tiende a 0.

En realidad no se precisa que sea autofunción, si no que esté en el conjunto $\mathcal{S}_\beta$ que se definió en el Lema 3.11.

Lemma 3.13. *Let* $w\colon \mathbb{T}_m \to \mathbb{R}$ *be the eigenfunction associated to* $\lambda_1(\beta)$ *given in Lemma 3.11. Then*

$$\liminf_{k\to\infty} \frac{w_k - w_{k+1}}{p_\beta^k} \ge \lambda_1.$$

Proof. Using that $w(x) = w_{|x|}$ for all $x \in \mathbb{T}_m$ and the fact that w is an eigenfunction, we have

$$0 < \lambda_1 w_k p_\beta^k = \beta(w_k - w_{k-1}) + (1-\beta)(w_k - w_{k+1}) \quad \forall k \ge 1.$$

So, since $w_k > w_{k+1}$ we have that

$$(w_{k-1} - w_k) < \frac{1-\beta}{\beta}(w_k - w_{k+1}) = p_\beta^{-1}(w_k - w_{k+1}) \quad \forall k \ge 1.$$

Finally, we have that $\lambda_1 = 1 - w_1 = w_0 - w_1 \le p_\beta^{-k}(w_k - w_{k+1})$ for all $k \ge 1$, and the result follows.

☐

3.1. **A linear algebra interlude.** Consider the following generalization for the approach of Lemma 3.6. Given $k \ge 1$, let α_k be the minimum real number such that

res-general

$$\alpha_k u(x) + \Delta_\beta u(x) = 0 \qquad \text{for } |x| \le k, \tag{3.17}$$

where $u\colon \mathbb{T}_m \to \mathbb{R}$ is a non-negative function such that $u(x) = u_{|x|}$ with $u_j = 0$ for all $j > k$. If such α_k exists, the argument of Lemma 3.6 can be easily adapted to show that $\lambda_1(\beta) \le \alpha_k$.

The equation (3.1) can be possed in terms of an eigenvalue problem of the tridiagonal matrix $A_k = (a_{i,j})$ defined in (1.1). In these terms, α_k is an eigenvalue of A_k and the values of the function u is given by a positive eigenvector related to it (if there exists).

First, we will show that A_k is diagonalizable and all eigenvalues are positive. Recall that $A_k = (a_{i,j})$ is $k+1$ square tridiagonal matrix where

matrix-AA

$$a_{i,i} = p_\beta^{-i}, \qquad a_{i,i-1} = -\beta p_\beta^{-i}, \qquad a_{0,1} = -1 \qquad a_{i,i+1} = -(1-\beta)p_\beta^{-i}. \tag{3.18}$$

Note that $a_{i,i+1} = a_{i+1,i}$ for $i = 1, \ldots, k-1$, so, A_k is almost a symmetric matrix. Using some results of matrix analysis we can prove that there exists a diagonal matrix D_k with positive entries on its diagonal, such that $B_k = D_k A_k D_k^{-1}$ is symmetric (see for instance [6, p. 232]). Thus, A_k is diagonalizable.

To show that the all eigenvalues are positive we apply the Gershgorin's theorem to the rows of A_k to conclude that the spectrum of A_k is included in $[0, +\infty)$. Finnaly, it is easy to check that A_k is invertible. These facts imply that the minumum eigenvalue of A_k is positive. We denote it by α_k and write also $\bar{u}_k$ for an associated eigenvector (in fact, α_k will be a simple eigenvalue of A_k).

Now, we prove that $\bar{u}_k \geq 0$. Since A_k is invertible, $1/\alpha$ is the maximum eigenvalue of A_k^{-1} and $\bar{u}$ is also the eigenvector related to it.

Recall some definitions from matrix analysis. A matrix with non-positive off-diagonal elements is called a $M-$matrix when all of its eigenvalues has a positive real part. A matrix is totally positive if all minors are positive. In particular, a totally positive matrix has all entries positives. The following Lemma clarifies the introduction of these definitions.

Lemma 3.14. *Let C be a non-singular $M-$matrix. Then C^{-1} has non-negative entries. Moreover, C^{-1} is totally positive if and only if C is tridiagonal.*

The first part of this Lemma is proved in [7, p. 115] and the second part is in [8].

Our matrix A_k defined in (3.1), is a non-singular $M-$matrix, and by the previous Lemma, all elements of A_k^{-1} are positive. Then, the Perron-Frobenius theorem applied to A_k^{-1} implies that the maximum eigenvalue of A_k^{-1} is simple and its related eigenvector related has all entries positive. It follows that $\bar{u}_k \geq 0$.

We proved that $\lambda_1 \leq \alpha_k$ for all k. The following Lemma shows that $(\alpha_k)_{k\geq 1}$ is a decreasing sequence.

Lemma 3.15. *Let α_k be the real number defined in* (3.1). *Then, $\alpha_{k+1} < \alpha_k$ for all $k \geq 1$.*

Proof. First, remember that B_k and B_{k+1} are symmetric matrices with the same eigenvalues of A_k and A_{k+1} respectively. Note that B_{k+1} can be written by blocks and using B_k as the principal $(k+1)\times(k+1)-$block, that is

$$B_{k+1} = \begin{pmatrix} B_k & y \\ y^\top & p_\beta^{-(k+1)} \end{pmatrix},$$

where $y = -(1-\beta)p^{-k}e_{k+1}$ with e_{k+1} the last vector of the canonical basis of $\mathbb{R}^{k+1}$ and $y^\top$ its transpose. By the Cauchy's interlacing theorem (see [6, p. 242]) we deduce that

$$\alpha_{k+1} \leq \alpha_k,$$

and using the fact that B_{k+1} is tridiagonal with $b_{i,i+1} \neq 0$ for all i, the inequality is strict (see also [6, p. 258]). ☐

Está claro que si $\bar{u}_k \in \mathbb{R}^{k+1}$ es el vector propio asociado a α_k, que es positivo, puedo suponer que sus coordenadas suman 1. Si puedo extraer una subsucesión que converja a $\bar{u}$, un vector infinito pero donde todas las coordenadas son positivas y tiende a 0, entonces tengo un autovector asociado a λ_1 (o debería) y se prueba que $\bar{\lambda}_1 = \inf_k \alpha_k = \lambda_1$.

Con un argumento diagonal tipo Arzela-Áscoli creo que sale que hay una subsucesión que converge, pero no sé cómo ver que es positiva.

`eq`

4. Evolution equation

Aún no miré este capítulo

Our first goal in this section is to show that, there is a unique solution of (1.1) when $f \in L^\infty(\mathbb{T}_m, \mathbb{R})$.

Theorem 4.1. *Let* $\beta \in [0, \frac{1}{2}]$ *and* $f \in L^\infty(\mathbb{T}_m)$. *Then there is a unique solution* $u \in L^\infty(\mathbb{T}_m \times [0,\infty))$ *of* (1.1).

Proof. Let's start observing that $w_1(x,t) = \|f\|_\infty$ and $w_2(x,t) = -\|f\|_\infty$ satisfy

$$w_2(x,t) \leq w_1(x,t) \text{ and } w_2(x,t) \leq K_f^\beta w_2(x,t)$$

for any $(x,t) \in \mathbb{T}_m \times [0,\infty)$. Then

$$\mathcal{S} \coloneqq \{w \in L^\infty(\mathbb{T}_m \times [0,\infty)) \colon w \leq w_1 \text{ and } w \leq K_f^\beta w \text{ in } \mathbb{T}_m \times [0,\infty)\} \neq \emptyset.$$

Therefore

$$u(x,t) \coloneqq \sup\{w(x,t) \colon w \in \mathcal{S}\} \quad \forall (x,t) \in \mathbb{T}_m \times [0,\infty]$$

is well-defined. Moreover

$$-w_2(x,t) \leq u(x,t) \leq w_1(x,t) \text{ and } u(x,t) \leq K_f^\beta u(x,t)$$

for any $(x,t) \in \mathbb{T}_m \times [0,\infty)$.

Let's show that $u(x,t) = K_f^\beta u(x,t)$ for any $(x,t) \in \mathbb{T}_m \times [0,\infty)$. By contradiction assume that there is $(x_0,t_0) \in \mathbb{T}_m \times [0,\infty)$ such that

$$u(x_0,t_0) < K_f^\beta u(x_0,t_0).$$

Then there is $\varepsilon > 0$ such that

$$u(x_0,t_0) + \varepsilon \leq K_f^\beta u(x_0,t_0).$$

We now define

$$\bar{u}(x,t) \coloneqq \begin{cases} u(x,t) & \text{if } (x,t) \neq (x_0,t_0), \\ u(x,t) + \varepsilon & \text{if } (x,t) = (x_0,t_0). \end{cases}$$

Then

$$u(x,t) \leq \bar{u}(x,t) \text{ and } \bar{u}(x,t) \leq K_f^\beta \bar{u}(x,t)$$

for any $(x,t) \in \mathbb{T}_m \times [0,\infty)$. Moreover, by Theorem 2.3, we have that

$$\bar{u}(x,t) \leq w_1(x,t)$$

for any $(x,t) \in \mathbb{T}_m \times [0,\infty)$. Then $\bar{u} \in \mathcal{S}$ which leads a contradiction. Therfore $u(x,t) = K_f^\beta u(x,t)$ for any $(x,t) \in \mathbb{T}_m \times [0,\infty)$, that is u is a solution of (1.1).

Finally, the uniqueness follows from Theorm 2.3. $\square$

To conclude this article, we observe that there is an exponential decay for the solution of the evolution problem.

`teo_evo` **Theorem 4.2.** *Let* $\beta \in (0,\frac{1}{2})$, $f \in L^\infty(\mathbb{T}_m)$ *and* u *be the solution of* (1.1). *Then*

$$|u(x,t)| \leq \|f\|_\infty e^{-\lambda_1(\beta)t}$$

where $\lambda_1(\beta)$ *is the principal eigenvalue of* Δ_β.

ACKNOWLEDGMENTS

L.D.P. and J.D.R. are partially supported by CONICET grant PIP GI 11220150100036CO (Argentina), PICT-2018-03183 (Argentina) and UBACyT grant 20020160100155BA (Argentina). N.F. is partially supported by Fondo Vaz Ferreira FVF-061-2021 (Uruguay). MathAmSud?

REFERENCES

`BNV` [1] H. Berestycki, L. Nirenberg, and S. R. S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains.* Comm. Pure Appl. Math. 47 (1994), no. 1, 47–92.

`DPFR1` [2] L. M. Del Pezzo, N. Frevenza and J. D. Rossi. *Dirichlet-to-Neumann maps on Trees.* Potential Anal 53, 1423–1447 (2020).

`DPMR1` [3] L. M. Del Pezzo, C. A. Mosquera and J.D. Rossi, *The unique continuation property for a nonlinear equation on trees*, J. Lond. Math. Soc. (2) 89 (2014), no. 2, 364–382.

`DPMR2` [4] L. M. Del Pezzo, C. A. Mosquera and J.D. Rossi, *Estimates for nonlinear harmonic measures on trees.* Bull. Braz. Math. Soc. (N.S.) 45 (2014), no. 3, 405–432.

`DPMR3` [5] L. M. Del Pezzo, C. A. Mosquera and J.D. Rossi, *Existence, uniqueness and decay rates for evolution equations on trees.* Port. Math. 71 (2014), no. 1, 63–77.

`HJ-MA` [6] R. A. Horn and C. R. Johnson. *Matrix analysis.* Second edition. Cambridge University Press, Cambridge, 2013. xviii+643 pp.

`HJ-TMA` [7] R. A. Horn and C. R. Johnson. *Topics in matrix analysis.* Cambridge University Press, Cambridge, 1991. xviii+607 pp.

`Pena` [8] J. M. Peña. *$M-$matrices whose inverses are totally positive.* Linear Algebra Appl. 221 (1995), 189?193.

LEANDRO M. DEL PEZZO
IMAS AND DEPARTAMENTO DE MATEMÁTICA, FCEYN, UNIVERSIDAD DE BUENOS AIRES,
PABELLON I, CIUDAD UNIVERSITARIA (1428), BUENOS AIRES, ARGENTINA.

NICOLÁS FREVENZA
INSTITUTO DE ESTADÍSTICA, FCEA, UNIVERSIDAD DE LA REPÚBLICA,
GONZALO RAMÍREZ 1926 (11200), MONTEVIDEO, URUGUAY.

Julio D. Rossi
Departamento de Matemática, FCEyN, Universidad de Buenos Aires,
Pabellon I, Ciudad Universitaria (1428), Buenos Aires, Argentina.

Email address: `ldpezzo@dm.uba.ar, nicolas.frevenza@fcea.edu.uy, jrossi@dm.uba.ar`