EXISTENCE RESULTS FOR A LOCAL-NONLOCAL SYSTEM WITH COMBINED CRITICAL SOBOLEV TERMS

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Abstract. In this paper, we study the existence and multiplicity of nontrivial solutions to a local–nonlocal system involving combined critical nonlinearities. Our results are obtained by decomposing the Nehari manifold and applying the mountain pass theorem of Ambrosetti and Rabinowitz.

Contents

1. Introduction and main results

In this article, we study the existence of positive solutions to the following system:

$$\begin{cases} -\Delta u + (-\Delta)^s u = |u|^{2^*-2}u + \dfrac{\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^{\beta}u + \lambda \dfrac{h(x)}{q} G_u(u,v), & x \in \Omega, \\ -\Delta v + (-\Delta)^s v = |v|^{2^*-2}v + \dfrac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \lambda \dfrac{h(x)}{q} G_v(u,v), & x \in \Omega, \\ u, v = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $s \in (0,1)$, $n \geq 3$, $\lambda > 0$, and the exponents satisfy

$$1 < q < 2, \qquad \alpha, \beta > 1, \qquad \alpha + \beta = 2^* := \frac{2n}{n-2}.$$

Throughout the paper, we assume that

$$G\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$$

is a homogeneous nonlinearity of degree $q \in (1,2)$, while

$$h\colon \Omega \to \mathbb{R}$$

is a positive bounded weight function. The precise assumptions on G and h will be stated later.

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Here,

$$(-\Delta)^s u(x) \coloneqq C(n,s)\,\text{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy,$$

denotes the fractional Laplacian, where P.V. stands for the Cauchy principal value.

The system above combines local and nonlocal diffusion through the mixed operator

$$-\Delta + (-\Delta)^s,$$

together with critical nonlinearities and a lower-order homogeneous perturbation. Problems involving mixed local–nonlocal operators have attracted increasing attention in recent years due to the rich variational structure generated by the interaction between the Laplacian and the fractional Laplacian. Besides their intrinsic mathematical interest, such operators also arise naturally in several applications involving anomalous diffusion, phase transition phenomena, optimization, image processing, and reaction–diffusion models.

1.1. **Context and previous results.** Problems involving critical Sobolev nonlinearities have been extensively studied in the framework of elliptic equations and systems. In the local setting, the seminal paper of Brezis and Nirenberg [10] showed that the lack of compactness associated with the critical Sobolev embedding plays a fundamental role in the existence theory.

Variational methods have since become one of the main tools in the study of critical elliptic problems. A major breakthrough was the concentration–compactness principle introduced by Lions [19, 20], which provides a powerful framework to overcome the loss of compactness caused by critical Sobolev embeddings.

The interaction between concave and convex nonlinearities produces a particularly rich variational geometry and has been extensively investigated since the pioneering work of Ambrosetti, Brezis and Cerami [2]. In this framework, Nehari manifold methods and fibering map techniques have proved to be extremely effective in obtaining multiplicity results for critical elliptic problems.

The influence of the topology of the domain on the existence and multiplicity of solutions was investigated in [6, 12], while multiplicity results based on critical point theory and Nehari manifold techniques can be found in [3, 14, 17]. We also refer to [22] for uniqueness and non-degeneracy results for least energy solutions of critical elliptic systems.

In the nonlocal setting, considerable attention has been devoted to equations and systems involving the fractional Laplacian. As in the local case, the main analytical difficulty comes from the lack of compactness of the critical Sobolev embedding

$$H^s(\mathbb{R}^n) \hookrightarrow L^{2_s^*}(\mathbb{R}^n), \qquad 2_s^* = \frac{2n}{n-2s},$$

which is typically handled through variational methods and concentration–compactness arguments [19, 20].

The purely fractional counterpart of system (1.1) was investigated in [5], where the authors established existence and multiplicity results for positive solutions involving homogeneous nonlinearities of the form $G(u,v)$. We also mention the fractional Brezis–Nirenberg type result obtained by Servadei and Valdinoci [23].

More recently, mixed local–nonlocal operators of the form

$$-\Delta + (-\Delta)^s$$

have also attracted increasing attention; see, for example, [7, 14].These operators combine features from both local and nonlocal equations and give rise to new phenomena that do not appear in the purely local or purely fractional settings. Mathematically, this combination introduces a competition between the short-range interactions governed by the classical Laplacian and the long-range effects driven by the fractional Laplacian, significantly affecting the asymptotic behavior of solutions depending on the scale.

One of the main features of problem (1.1) is that, although the operator contains a nonlocal component, the critical exponent remains the classical Sobolev exponent

$$2^* = \frac{2n}{n-2},$$

rather than the fractional critical exponent. This produces a variational structure substantially different from the purely fractional case and introduces additional difficulties in the compactness analysis.

While the aforementioned works provide a solid background for critical equations, the study of systems with combined local-nonlocal critical terms remains widely open. Motivated by this gap, the main goal of this paper is to investigate the existence and multiplicity of positive solutions for the system (1.1).

1.2. **Main contribution.** In order to state our main results, we begin by introducing the variational framework and the algebraic properties of the nonlinearities. Our approach combines variational methods, concentration–compactness arguments, and a detailed analysis of the geometry of the associated Nehari manifold.

The proof is not a straightforward adaptation of the arguments developed for the purely fractional framework in [5]. The main difficulty stems from the behavior of the associated Sobolev constants and, in particular, from the lack of compactness phenomena specific to mixed local–nonlocal operators.

In the purely fractional setting, the sharp Sobolev constant

$$\mathfrak{A}_{n,\alpha,\beta} := \inf \left\{ \frac{[u]_s^2 + [v]_s^2}{\left(\int_{\mathbb{R}^n} |u|^\alpha |v|^\beta \, dx \right)^{2/2^*}} : u, v \in H^s(\mathbb{R}^n) \setminus \{0\} \right\}$$

is closely related to the fractional Sobolev constant

$$\mathfrak{S}_n := \inf \left\{ \frac{[u]_s^2}{\left(\int_{\mathbb{R}^n} |u|^{2^*} \, dx \right)^{2/2^*}} : u \in H^s(\mathbb{R}^n) \setminus \{0\} \right\},$$

and the relation

$$\mathfrak{A}_{n,\alpha,\beta} = \left[\left(\frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}} \right] \mathfrak{S}_n$$

plays a crucial role in the variational analysis carried out in [5]. Moreover, it is known from [13] that the constant $\mathfrak{S}_n$ is attained.

In contrast, for the mixed framework considered here, the corresponding Sobolev constant

$$\mathcal{A}_{n,\alpha,\beta} := \inf \left\{ \frac{\|u\|_{s,2}^2 + \|v\|_{s,2}^2}{\left(\int_{\Omega} |u|^\alpha |v|^\beta \, dx \right)^{2/2^*}} : u, v \in D^{1,2}(\mathbb{R}^n) \setminus \{0\} \right\}$$

also admits an analogous characterization in terms of the scalar Sobolev constant, see Lemma 2.2 below.

However, a major difference with the purely fractional setting is that the mixed Sobolev constant

$$\mathcal{S}_{n,s} := \inf \left\{ \frac{\|u\|_{s,2}^2}{\left(\int_{\mathbb{R}^n} |u|^{2^*} \, dx \right)^{2/2^*}} : u \in D^{1,2}(\mathbb{R}^n) \setminus \{0\} \right\}$$

is never attained in

$$D^{1,2}(\mathbb{R}^n) := \left\{ u \in L^{2^*}(\mathbb{R}^n) \colon |\nabla u| \in L^2(\mathbb{R}^n) \text{ and } [u]_s < \infty \right\},$$

see [7]. This lack of attainment produces substantial additional difficulties in the variational analysis and forces us to develop several arguments that are specific to the mixed framework.

Here,

$$[u]_s^2 := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy,$$

and

$$\|u\|_{s,2}^2 \coloneqq [u]_s^2 + \int_{\mathbb{R}^n} |\nabla u|^2\, dx.$$

A crucial ingredient in our approach is the decomposition of the Nehari manifold into two disjoint components associated with different variational structures. This decomposition allows us to obtain two distinct positive solutions corresponding to different energy levels.

In order to state our main result, we introduce the assumptions on the nonlinear term and the weight function.

Throughout the paper, we assume that the nonlinearities G and h satisfy the following assumptions:

(H1) $G \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ satisfies the homogeneity condition

$$G(tu, tv) = t^q G(u, v) \quad \text{for all } u, v \in \mathbb{R}^+ \text{ and } t > 0.$$

(H2) $G(u, 0) = G(0, v) = G_u(0, 1) = G_v(1, 0) = 0$, for every $u, v \in \mathbb{R}^+$.

(H3) The function $h \in C(\overline{\Omega})$ is positive and bounded, with

$$\hat{h} \coloneqq \min\{h(x) : x \in \overline{\Omega}\} > 0.$$

We can now state our main result.

Theorem 1.1. *Let $n \geq 3$, $q \in (1, 2)$ and assume that conditions **(H1)**–**(H3)** hold. Then there exists $\lambda_0 > 0$ such that, for every $\lambda \in (0, \lambda_0)$, system (1.1) admits at least two positive solutions.*

For further details concerning these Sobolev constants and the mixed framework, we refer the reader to Subsection 2.2.

The remainder of the paper is organized as follows. In Section 2, we collect several preliminary results and introduce the functional framework associated with the mixed operator. In Section 3, we analyze the geometry of the corresponding Nehari manifold and establish the variational setting of the problem. Finally, in Section 4, we prove Theorem 1.1.

2. Preliminaries

In this section, we introduce the functional framework associated with the mixed operator

$$-\Delta + (-\Delta)^s,$$

and collect several preliminary results that will be used throughout the paper. In particular, we analyze some variational properties of the corresponding Sobolev constants and establish auxiliary results related to the geometry of the Nehari manifold.

Throughout the paper, we assume that $s \in (0, 1)$ is fixed, $n \geq 3$, and $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. Moreover, the parameters $\lambda > 0$, $q \in (1, 2)$, and $\alpha, \beta > 1$ satisfy

$$\alpha + \beta = 2^*, \qquad 2^* = \frac{2n}{n-2}.$$

We also assume that the nonlinearities G and h satisfy conditions (**H1**)–(**H3**).

The following example shows that assumptions (**H1**)–(**H2**) are nonempty and provide a concrete class of admissible nonlinearities.

Example 2.1. *Let $q \in (1, 2)$ and define*

$$G(u, v) \coloneqq \frac{(uv)^{\frac{q}{2}+1}}{u^2 + v^2}, \qquad (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

*with the convention $G(u, 0) = G(0, v) = 0$. Then $G \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ satisfies assumptions **(H1)**–**(H2)**. Indeed, G is positively homogeneous of degree q, vanishes on the coordinate axes, and*

$$G_u(0, 1) = G_v(1, 0) = 0.$$

2.1. **Functional setting.** The fractional Sobolev space $H^s(\Omega)$ is defined by

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \colon [u]_{s,\Omega} < +\infty \right\},$$

where

$$[u]^2_{s,\Omega} := \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dydx.$$

We recall some fundamental properties of $H^s(\Omega)$:

- The space $H^s(\Omega)$ is a Hilbert space equipped with the inner product:
$$\langle u, v\rangle_{s,\Omega} := \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx.$$
- The space $C_0^\infty(\Omega)$ is a subspace of $H^s(\Omega)$. Moreover, $C_0^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.
- If $\Omega = \mathbb{R}^n$ or Ω has bounded Lipschitz boundary, then
$$H^1(\Omega) \hookrightarrow H^s(\Omega)$$
continuously. In particular, there exists $C = C(n, s) > 0$ such that

$$[u]^2_{s,\Omega} \le C\|u\|^2_{H^1(\Omega)}, \qquad \forall u \in H^1(\Omega). \tag{2.1}$$

We now introduce the functional framework associated with the mixed local–nonlocal operator. Define

$$\mathcal{X}^{1,2}(\Omega)$$

as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{s,2} := \int_{\mathbb{R}^n} |\nabla u|^2\, dx + \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dy\, dx.$$

This norm is induced by the inner product

$$\langle u, v\rangle_{s,2} := \int_{\mathbb{R}^n} \nabla u\cdot\nabla v\, dx + \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dy\, dx,$$

and therefore $\mathcal{X}^{1,2}(\Omega)$ is a Hilbert space.

By Poincaré's inequality and (2.1), there exists a constant $\Gamma = \Gamma(n, s, \Omega) > 0$ such that

$$\Gamma^{-1}\|u\|_{H^1(\mathbb{R}^n)} \le \|u\|_{s,2} \le \Gamma\|u\|_{H^1(\mathbb{R}^n)} \quad \forall u \in C_0^\infty(\Omega).$$

Consequently,

$$\begin{aligned}\mathcal{X}^{1,2}(\Omega) &= \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\mathbb{R}^n)}}\\ &= \left\{ u \in H^1(\mathbb{R}^n)\colon u|_\Omega \in H^1_0(\Omega) \text{ and } u = 0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega \right\}.\end{aligned}$$

On the other hand, if $\Omega = \mathbb{R}^n$ then

$$\mathcal{X}^{1,2}(\mathbb{R}^n) = D^{1,2}(\mathbb{R}^n).$$

For further details, we refer the reader to [7].

Given a function $u\colon \Omega \to \mathbb{R}$, we define its positive and negative parts by

$$u_+(x) := \max\{u(x), 0\} \quad \text{and} \quad u_-(x) := \max\{-u(x), 0\}.$$

Finally, let us introduce the function space

$$\mathbb{H}^s(\Omega) := \mathcal{X}^{1,2}(\Omega) \times \mathcal{X}^{1,2}(\Omega)$$

endowed with the norm

$$|||(u, v)|||^2 := \|u\|^2_{s,2} + \|v\|^2_{s,2}.$$

2.2. **The mixed Sobolev constant.** Let $\mathcal{S}_n$ denote the best Sobolev constant associated with the Sobolev embedding

$$D^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)\}$$

defined by

$$\mathcal{S}_n \coloneqq \inf\left\{ \frac{\displaystyle\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^n} |u|^{2^*} dx\right)^{\frac{2}{2^*}}} : u, v \in D^{1,2}(\mathbb{R}^n) \setminus \{0\} \right\}. \tag{2.2}$$

The infimum is attained by the family of functions

$$\mathcal{U}_{\sigma,x_0}(x) = \sigma^{\frac{2-n}{2}} \omega\left(\frac{x - x_0}{\sigma}\right), \quad \sigma > 0,\ x_0 \in \mathbb{R}^n, \tag{2.3}$$

where

$$\omega(x) \coloneqq c(1 + |x|^2)^{\frac{2-n}{2}},$$

and $c > 0$ is a normalization constant chosen so that $\|\omega\|_{L^{2^*}(\mathbb{R}^n)} = 1$.

On the other hand, it was proved in [7] that the optimal mixed Sobolev constant

$$\mathcal{S}_{n,s} \coloneqq \inf\left\{ \frac{\|u\|_{s,2}^2}{\left(\displaystyle\int_{\mathbb{R}^n} |u|^{2^*} dx\right)^{2/2^*}} : u, v \in D^{1,2}(\mathbb{R}^n) \setminus \{0\} \right\}$$

coincides with $\mathcal{S}_n$, namely,

$$\mathcal{S}_{n,s} = \mathcal{S}_n.$$

However, unlike the fractional case, the infimum defining $\mathcal{S}_{n,s}$ is never attained in $D^{1,2}(\mathbb{R}^n)$. Nevertheless, [7] shows that $\mathcal{S}_{n,s}$ is asymptotically achieved in the sense that

$$\lim_{\sigma\to\infty} \|\mathcal{U}_{\sigma,x_0}\|_{s,2}^2 = \|\nabla\omega\|_{L^2(\mathbb{R}^n)}^2 = \mathcal{S}_n.$$

For $\alpha, \beta > 1$ satisfying

$$\alpha + \beta = 2^*,$$

we introduce the Sobolev-type constant

$$\mathcal{A}_{n,\alpha,\beta} \coloneqq \inf\left\{ \frac{|||(u,v)|||^2}{\left(\displaystyle\int_{\mathbb{R}^n} \left[|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta\right] dx\right)^{2/2^*}} : (u,v) \in \mathbb{H}^s(\mathbb{R}^n) \setminus \{(0,0)\} \right\}. \tag{2.4}$$

As shown in the proof of [22, Lemma 2.1], the constant $\mathcal{A}_{n,\alpha,\beta}$ is attained by functions of the form

$$(\mathcal{U}_{\sigma,x_0}, t_*\mathcal{U}_{\sigma,x_0}),$$

where $x_0 \in \mathbb{R}^n$, $\sigma > 0$, and $\mathcal{U}_{\sigma,x_0}$ is given by (2.3). The parameter $t_* > 0$ is a solution of

$$2^* + \alpha t^\beta - \beta t^{\beta-2} - 2^* t^{2^*-2} = 0, \quad t \geq 0. \tag{2.5}$$

Define the function $\mathfrak{f}\colon [0,\infty) \to \mathbb{R}$

$$\mathfrak{f}(t) \coloneqq \frac{1 + t^2}{(1 + t^\beta + t^{2^*})^{2/2^*}}.$$

Lemma 2.2. *The function $\mathfrak{f}$ attains its minimum at $t_* > 0$, namely,*

$$\mathfrak{f}(t_*) = \min_{t\geq 0} \mathfrak{f}(t) > 0.$$

Furthermore, the relationship between $\mathcal{S}_n$ and $\mathcal{A}_{n,\alpha,\beta}$ is given by

$$\mathcal{A}_{n,\alpha,\beta} = \mathfrak{f}(t_*)\mathcal{S}_n. \tag{2.6}$$

Proof. First, note that

$$\lim_{t\to 0^+} \mathfrak{f}(t) = 1 = \lim_{t\to +\infty} \mathfrak{f}(t),$$

which implies that the minimum of $\mathfrak{f}$ is attained at some $t_* \in (0,\infty)$. A direct computation shows that there exists $C > 0$ such that

$$0 < C \le \mathfrak{f}(t_*) = \min_{t\ge 0} \mathfrak{f}(t) \le 1, \quad t_* \in (0,\infty).$$

Since $\mathfrak{f}'(t_*) = 0$, it follows that t_* satisfies (2.5).

To establish (2.6), we follow the ideas of [1]. Let $(w_k) \subset D^{1,2}(\mathbb{R}^n) \setminus \{0\}$ be a minimizing sequence for $\mathcal{S}_n$. For $\mu_1, \mu_2 > 0$ (to be chosen later), set $u_k = \mu_1 w_k$ and $v_k = \mu_2 w_k$. Substituting into (2.4), we obtain

$$\begin{aligned}\mathcal{A}_{n,\alpha,\beta} &\le \frac{|||(\mu_1 w_k, \mu_2 w_k)|||^2}{\left(\int_{\mathbb{R}^n} \left(|\mu_1 w_k|^{2^*} + |\mu_2 w_k|^{2^*} + |\mu_1 w_k|^{\alpha} |\mu_2 w_k|^{\beta}\right) dx\right)^{2/2^*}} \\ &= \frac{\mu_1^2 + \mu_2^2}{(\mu_1^{2^*} + \mu_2^{2^*} + \mu_1^{\alpha}\mu_2^{\beta})^{2/2^*}} \frac{\|w_k\|_{s,2}^2}{\left(\int_{\mathbb{R}^n} |w_k|^{2^*} dx\right)^{2/2^*}} \\ &= \mathfrak{f}\left(\frac{\mu_2}{\mu_1}\right) \frac{\|w_k\|_{s,2}^2}{\left(\int_{\mathbb{R}^n} |w_k|^{2^*} dx\right)^{2/2^*}}.\end{aligned}$$

Choosing μ_1 and μ_2 such that $\frac{\mu_2}{\mu_1} = t_*$ and letting $k \to +\infty$, we deduce

$$\mathfrak{f}(t_*)\mathcal{S}_n \ge \mathcal{A}_{n,\alpha,\beta}. \tag{2.7}$$

Conversely, let $\{(u_k, v_k)\} \subset \mathbb{H}^s(\mathbb{R}^n) \setminus \{(0,0)\}$ be a minimizing sequence for $\mathcal{A}_{n,\alpha,\beta}$. Set $z_k = \mu_k v_k$ with $\mu_k > 0$ chosen so that

$$\int_{\mathbb{R}^n} |u_k|^{2^*} dx = \int_{\mathbb{R}^n} |z_k|^{2^*} dx.$$

By Young's inequality,

$$\int_{\mathbb{R}^n} |u_k|^{\alpha} |z_k|^{\beta} dx \le \frac{\alpha}{2^*} \int_{\mathbb{R}^n} |u_k|^{2^*} dx + \frac{\beta}{2^*} \int_{\mathbb{R}^n} |z_k|^{2^*} dx = \int_{\mathbb{R}^n} |u_k|^{2^*} dx = \int_{\mathbb{R}^n} |z_k|^{2^*} dx.$$

Thus,

$$\begin{aligned}&\frac{|||(u_k, v_k)|||^2}{\left(\int_{\mathbb{R}^n} \left(|u_k|^{2^*} + |v_k|^{2^*} + |u_k|^{\alpha}|v_k|^{\beta}\right) dx\right)^{2/2^*}} \\ &\ge \frac{\|u_k\|_{s,2}^2}{\left(1 + \frac{1}{\mu_k^{2^*}} + \frac{1}{\mu_k^{\beta}}\right)^{2/2^*} \left(\int_{\mathbb{R}^n} |u_k|^{2^*} dx\right)^{2/2^*}} + \frac{\frac{1}{\mu_k^2}\|z_k\|_{s,2}^2}{\left(1 + \frac{1}{\mu_k^{2^*}} + \frac{1}{\mu_k^{\beta}}\right)^{2/2^*} \left(\int_{\mathbb{R}^n} |z_k|^{2^*} dx\right)^{2/2^*}} \\ &\ge \mathfrak{f}\left(\frac{1}{\mu_k}\right) \mathcal{S}_n \\ &\ge \mathfrak{f}(t_*)\mathcal{S}_n.\end{aligned}$$

Taking $k \to \infty$, we conclude

$$\mathcal{A}_{n,\alpha,\beta} \ge \mathfrak{f}(t_*)\mathcal{S}_n. \tag{2.8}$$

Combining (2.7) and (2.8) yield (2.6). □

2.3. **Properties of the function G.** The next lemma follows from standard arguments, and we omit its proof. See for instance [1, Remark 5 (iv)].

Lemma 2.3. *Let $G \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ be positively homogeneous of degree q. Then $G_u, G_v \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ are positively homogeneous of degree $q-1$.*

We now extend G to the whole space $\mathbb{R}^2$. Under hypothesis (**H2**), the definition

$$\widetilde{G}(u,v) := G(|u|,|v|)$$

yields a C^1extension. For simplicity, we continue to denote this extension by G.

Remark 2.4. *By assumption (**H1**), the Euler identity*

$$(u,v)\cdot\nabla G(u,v) = qG(u,v) \quad \forall u,v \in \mathbb{R}$$

holds. Moreover, there exists a constant $\mathcal{K}_1 > 0$ such that

$$G(u,v) \le \mathcal{K}_1 \left(|u|^2 + |v|^2\right)^{q/2} \quad \forall u,v \in \mathbb{R}^+.$$

Let

$$\overline{G} := \max\{G(u,v) \colon u,v \in \mathbb{R}, |u|^q + |v|^q = 1\} > 0,$$

we obtain the upper bound

$$G(u,v) \le \overline{G}(|u|^q + |v|^q), \ \ \forall (u,v) \in \mathbb{R}\times\mathbb{R}. \tag{2.9}$$

As a consequence of Lemma 2.3, there exists a positive constant $\mathcal{M}$ such that, for all $u,v \in \mathbb{R}^+$, the following estimates hold:

$$|G_u(u,v)| \le \mathcal{M}(|u|^{q-1} + |v|^{q-1}), \tag{2.10}$$

and

$$|G_v(u,v)| \le \mathcal{M}(|u|^{q-1} + |v|^{q-1}). \tag{2.11}$$

2.4. **Palais-Smale condition.** Before proving the Palais-Smale condition for suitable values of c, we first introduce some notation and establish several technical lemmas.

We define

$$Q_\lambda(u,v) := \int_\Omega \lambda h(x) G(u,v) dx \quad \text{and} \quad Q_\theta := \left(\int_\Omega h^\theta(x) dx\right)^{1/\theta},$$

with $\theta = \dfrac{2^*}{2^*-q}$ (see (**H3**)).

The C^1 functional $J_\lambda \colon \mathbb{H}^s(\Omega) \to \mathbb{R}$ associated with problem (1.1) is given by

$$J_\lambda(u,v) := \frac{1}{2}|||(u,v)|||^2 - \frac{1}{q}Q_\lambda(u,v) - \frac{1}{2^*}\int_\Omega \left(|u|^{2^*} + |v|^{2^*}\right) dx - \frac{1}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^\beta dx.$$

The Fréchet derivative of J_λ is

$$\begin{aligned} \langle J'_\lambda(u,v), (\varphi,\phi)\rangle =& \langle u,\varphi\rangle_{s,2} + \langle v,\phi\rangle_{s,2} \\ &- \frac{\lambda}{q}\int_\Omega h(x)\nabla G(u,v)\cdot(\varphi,\phi) dx - \int_\Omega \left(|u|^{2^*-2}u\varphi + |v|^{2^*-2}v\phi\right) dx \\ &- \frac{\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^\beta \varphi dx - \frac{\beta}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^{\beta-2} v\phi dx. \end{aligned}$$

We recall that a sequence $\{(u_k, v_k)\}$ is called a Palais-Smale sequence at the level c (or a $(PS)_c$ sequence) for the functional J_λ if

$$J_\lambda(u_k,v_k) \to c \quad \text{and} \quad J'_\lambda(u_k,v_k) \to 0 \ \text{ as } k \to \infty. \tag{2.12}$$

Furthermore, if every $(PS)_c$ sequence $\{(u_k,v_k)\}$ admits a convergent subsequence, we say that J_λ satisfies the $(PS)_c$ condition.

To establish the $(PS)_c$ condition for a suitable range of c, we first study the gradient component of the norm $\|\cdot\|_{s,2}$.

Lemma 2.5. *Let $\{(u_k, v_k)\}_{k\in\mathbb{N}} \subset \mathbb{H}^s(\Omega)$ be a bounded sequence satisfying (2.12) for some $c \in \mathbb{R}$. Then, up to a subsequence, $\nabla u_k(x) \to \nabla u(x)$ and $\nabla v_k(x) \to \nabla v(x)$ a.e. in Ω as $k \to \infty$.*

Proof. To prove this result, we adapt some ideas from [14]. Since $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{H}^s(\Omega)$, there exists $(u, v) \in \mathbb{H}^s(\Omega)$ such that, up to a subsequence,

$$\begin{aligned}
&(u_k, v_k) \rightharpoonup (u, v) \ \text{ in } \mathbb{H}^s(\Omega), \text{ and } \quad (\nabla u_k, \nabla v_k) \rightharpoonup (\nabla u, \nabla v) \quad \text{in } [L^2(\Omega)]^n \times [L^2(\Omega)]^n,\\
&|u_k(x)| \le h_1(x) \quad \text{and} \quad |v_k(x)| \le h_2(x) \quad \text{a.e. in } \Omega, \text{ with } h_1, h_2 \in L^2(\Omega),\\
&u_k \to u \quad \text{and} \quad v_k \to v \quad \text{ in } L^p(\Omega), \text{ with } p \in [1, 2^*),\\
&u_k(x) \to u(x) \quad \text{and} \quad v_k(x) \to v(x) \quad \text{a.e. in } \Omega,
\end{aligned} \tag{2.13}$$

as $k \to \infty$.

For any $\kappa \in \mathbb{N}$, let $T_\kappa \colon \mathbb{R} \to \mathbb{R}$ be the truncation function defined by

$$T_\kappa(t) := \begin{cases} t, & \text{if } |t| \le \kappa,\\ \kappa \dfrac{t}{|t|}, & \text{if } |t| > \kappa. \end{cases}$$

Let $\kappa \in \mathbb{N}$ be fixed. Since $\{(T_\kappa(u_k - u), T_\kappa(v_k - v))\}_{k\in\mathbb{N}}$ is bounded, by (2.12) we have

$$\begin{aligned}
o_k(1) &= \langle J'_\lambda(u_k, v_k), (T_\kappa(u_k - u), T_\kappa(v_k - v))\rangle\\
&= \langle (u_k, v_k), (T_\kappa(u_k - u), T_\kappa(v_k - v))\rangle_{s,2}\\
&\quad - \frac{\lambda}{q}\int_\Omega h(x)\nabla G(u_k, v_k)\cdot(T_\kappa(u_k - u), T_\kappa(v_k - v))dx\\
&\quad - \int_\Omega \left(|u_k|^{2^*-2}u_k T_\kappa(u_k - u) + |v_k|^{2^*-2}v_k T_\kappa(v_k - v)\right)dx\\
&\quad - \frac{\alpha}{\alpha+\beta}\int_\Omega |u_k|^{\alpha-2}u_k|v_k|^\beta T_\kappa(u_k - u)dx\\
&\quad - \frac{\beta}{\alpha+\beta}\int_\Omega |u_k|^{\alpha}|v_k|^{\beta-2}v_k T_\kappa(v_k - v)dx,
\end{aligned} \tag{2.14}$$

as $k \to \infty$. By Hölder's inequality and (2.13), we obtain

$$\lim_{k\to\infty}\langle (u, v), (T_\kappa(u_k - u), T_\kappa(v_k - v))\rangle_{s,2} = 0. \tag{2.15}$$

By the boundedness of $\{(u_k, v_k)\}_{k\in\mathbb{N}}$, we have

$$\begin{aligned}
&\left|\int_\Omega \left(|u_k|^{2^*-2}u_k T_\kappa(u_k - u) + |v_k|^{2^*-2}v_k T_\kappa(v_k - v)\right)dx\right| \le\\
&\qquad\qquad \le \kappa\int_\Omega \left(|u_k|^{2^*-1} + |v_k|^{2^*-1}\right)dx \le C\kappa,
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
&\left|\frac{\alpha}{\alpha+\beta}\int_\Omega |u_k|^{\alpha-2}u_k|v_k|^\beta T_\kappa(u_k - u)dx + \frac{\beta}{\alpha+\beta}\int_\Omega |u_k|^\alpha|v_k|^{\beta-2}v_k T_\kappa(v_k - v)dx\right|\\
&\qquad \le \kappa\int_\Omega \left(|u_k|^{\alpha-1}|v_k|^\beta + |u_k|^\alpha|v_k|^{\beta-1}\right)dx\\
&\qquad \le \kappa\left(\|u_k\|^{\alpha-1}_{L^{2^*}(\Omega)}\|v_k\|^\beta_{L^{2^*}(\Omega)} + \|u_k\|^\alpha_{L^{2^*}(\Omega)}\|v_k\|^{\beta-1}_{L^{2^*}(\Omega)}\right)\\
&\qquad \le C\kappa,
\end{aligned} \tag{2.17}$$

where C is a positive constant independent of k and κ. In addition, by (2.10) and (2.11), we get

$$\begin{aligned}
&\left|\frac{\lambda}{q}\int_\Omega h(x)\nabla G(u_k, v_k)\cdot(T_\kappa(u_k - u), T_\kappa(v_k - v))dx\right| \le\\
&\qquad\qquad \le \kappa\mathcal{M}\frac{\lambda}{q}\|h\|_{L^\infty(\Omega)}\int_\Omega \left(|u_k|^{q-1} + |v_k|^{q-1}\right)dx \le C\kappa.
\end{aligned} \tag{2.18}$$

Here also, C is a positive constant independent of k and κ.

By (2.14), (2.15), (2.16), (2.17), and (2.18), we get

$$\begin{aligned}
&\limsup_{k\to\infty}\langle (u_k,v_k),(T_\kappa(u_k-u),T_\kappa(v_k-v))\rangle_{s,2} = \\
&\qquad = \limsup_{k\to\infty}\Big[\frac{\lambda}{q}\int_\Omega h(x)\nabla G(u_k,v_k)\cdot(T_\kappa(u_k-u),T_\kappa(v_k-v))dx \\
&\qquad\quad + \int_\Omega \left(|u_k|^{2^*-2}u_kT_\kappa(u_k-u)+|v_k|^{2^*-2}v_kT_\kappa(v_k-v)\right)dx \\
&\qquad\quad + \frac{\alpha}{\alpha+\beta}\int_\Omega |u_k|^{\alpha-2}u_k|v_k|^{\beta}T_\kappa(u_k-u)dx \\
&\qquad\quad + \frac{\beta}{\alpha+\beta}\int_\Omega |u_k|^{\alpha}|v_k|^{\beta-2}v_kT_\kappa(v_k-v)dx\Big] \\
&\qquad \le C\kappa.
\end{aligned}$$

From this point on, the proof proceeds as in [14, Proof of Lemma 2.2]. □

Lemma 2.6. *Let $\{(u_k,v_k)\}\subset\mathbb{H}^s(\Omega)$ be a $(PS)_c$ sequence for J_λ such that $(u_k,v_k)\rightharpoonup(u,v)$ in $\mathbb{H}^s(\Omega)$. Then $J'_\lambda(u,v)=0$ and $J_\lambda(u,v)\ge\mathcal{H}(z_*,z_*)$, where*

$$\begin{aligned}
&\mathcal{H}(z_*,z_*) = \inf_{z_1,z_2\ge 0}\mathcal{H}(z_1,z_2)<0,\\
&\mathcal{H}(z_1,z_2) := \left(\frac{1}{2}-\frac{1}{2^*}\right)(z_1^2+z_2^2)-\left(\frac{1}{q}-\frac{1}{2^*}\right)\lambda\overline{G}Q_\theta\mathcal{S}_n^{-\frac{q}{2}}(z_1^q+z_2^q)\\
&z_* := \left(\frac{2^*-q}{2^*-2}\lambda\overline{G}Q_\theta\mathcal{S}_n^{-\frac{q}{2}}\right)^{\frac{1}{2-q}}.
\end{aligned}$$

Proof. Since $\{(u_k,v_k)\}\subset\mathbb{H}^s(\Omega)$ is a $(PS)_c$ sequence for J_λ and $(u_k,v_k)\rightharpoonup(u,v)$ in $\mathbb{H}^s(\Omega)$, it follows from standard arguments that $J'_\lambda(u,v)=0$.

Using hypothesis (**H2**) and the fact that $J'_\lambda(u,v)=0$, we obtain:

$$|||(u,v)|||^2 - Q_\lambda(u,v) = \int_\Omega\left(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta\right)dx.$$

Consequently,

$$J_\lambda(u,v) = \left(\frac{1}{2}-\frac{1}{2^*}\right)|||(u,v)|||^2-\left(\frac{1}{q}-\frac{1}{2^*}\right)Q_\lambda(u,v).$$

By (**H3**), inequality (2.9), and the Hölder inequality, we have:

$$\begin{aligned}
Q_\lambda(u,v) &\le \lambda\overline{G}\int_\Omega h(x)\left(|u|^q+|v|^q\right)dx\\
&\le \lambda\overline{G}\left(\int_\Omega h^\theta(x)dx\right)^{1/\theta}\left[\left(\int_\Omega|u|^{2^*}dx\right)^{\frac{q}{2^*}}+\left(\int_\Omega|v|^{2^*}dx\right)^{\frac{q}{2^*}}\right]\\
&\le \lambda\overline{G}Q_\theta\mathcal{S}_n^{-\frac{q}{2}}\left(\|u\|_{s,2}^q+\|v\|_{s,2}^q\right).
\end{aligned}\tag{2.19}$$

Combining these results, we deduce:

$$\begin{aligned}
J_\lambda(u,v) &= \left(\frac{1}{2}-\frac{1}{2^*}\right)|||(u,v)|||^2-\left(\frac{1}{q}-\frac{1}{2^*}\right)Q_\lambda(u,v)\\
&\ge \left(\frac{1}{2}-\frac{1}{2^*}\right)|||(u,v)|||^2-\left(\frac{1}{q}-\frac{1}{2^*}\right)\lambda\overline{G}Q_\theta\mathcal{S}_n^{-\frac{q}{2}}\left(\|u\|_{s,2}^q+\|v\|_{s,2}^q\right)\\
&\ge \mathcal{H}(z_*,z_*).
\end{aligned}$$

This completes the proof. □

Using **(H3)** and the Sobolev Embedding Theorem, we obtain the following result.

Lemma 2.7. *Suppose that* $(u_k, v_k) \rightharpoonup (u, v)$ *in* $\mathbb{H}^s(\Omega)$. *Then*

$$\int_{\mathbb{R}^n} h(x)|u_k - u|^q dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^n} h(x)|v_k - v|^q dx \to 0 \ \text{ as } k \to \infty.$$

The next result shows that any $(PS)_c$ sequence for J_λ is bounded in $\mathbb{H}^s(\Omega)$.

Lemma 2.8. *Let* $\{(u_k, v_k)\}_k \subset \mathbb{H}^s(\Omega)$ *be a* $(PS)_c$ *sequence for* J_λ. *Then* $\{(u_k, v_k)\}_k$ *is bounded in* $\mathbb{H}^s(\Omega)$.

Proof. Assume, by contradiction, that $|||(u_k, v_k)||| \to +\infty$. Define the normalized sequence

$$(\tilde{u}_k, \tilde{v}_k) = \left(\frac{u_k}{|||(u_k, v_k)|||}, \frac{v_k}{|||(u_k, v_k)|||} \right).$$

Since $|||(\tilde{u}_k, \tilde{v}_k)||| = 1$, there exists $(\tilde{u}, \tilde{v}) \in \mathbb{H}^s(\Omega)$ such that $(\tilde{u}_k, \tilde{v}_k) \rightharpoonup (\tilde{u}, \tilde{v})$ in $\mathbb{H}^s(\Omega)$.

On the other hand, by Remark 2.4, we have

$$\begin{aligned} Q_\lambda(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v}) &= \int_\Omega \lambda h(x) G(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v}) dx \\ &\le \lambda \overline{G} \int_\Omega h(x) \left(|\tilde{u}_k - \tilde{u}|^q + |\tilde{v}_k - \tilde{v}|^q \right) dx. \end{aligned}$$

Applying Lemma 2.7, we obtain:

$$Q_\lambda(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v}) \to 0 \ \text{ as } \ k \to \infty.$$

Consequently,

$$Q_\lambda(\tilde{u}_k, \tilde{v}_k) = Q_\lambda(\tilde{u}, \tilde{v}) + o_k(1). \tag{2.20}$$

Since $\{(u_k, v_k)\}_k$ is a $(PS)_c$ sequence for J_λ, it satisfies:

$$\begin{aligned} \frac{1}{2}|||(\tilde{u}_k, \tilde{v}_k)|||^2 - \frac{1}{2^*}|||(u_k, v_k)|||^{2^*-2} \int_\Omega \left(|\tilde{u}_k|^{2^*} + |\tilde{v}_k|^{2^*} + |\tilde{u}_k|^\alpha |\tilde{v}_k|^\beta \right) dx \\ - \frac{1}{q}|||(u_k, v_k)|||^{q-2} Q_\lambda(\tilde{u}_k, \tilde{v}_k) = o_k(1), \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} |||(\tilde{u}_k, \tilde{v}_k)|||^2 - |||(u_k, v_k)|||^{2^*-2} \int_\Omega \left(|\tilde{u}_k|^{2^*} + |\tilde{v}_k|^{2^*} + |\tilde{u}_k|^\alpha |\tilde{v}_k|^\beta \right) dx \\ - |||(u_k, v_k)|||^{q-2} Q_\lambda(\tilde{u}_k, \tilde{v}_k) = o_k(1). \end{aligned} \tag{2.22}$$

Combining (2.20), (2.21), (2.22), and using $q \in (1, 2)$ and $|||(u_k, v_k)||| \to +\infty$, we deduce:

$$|||(\tilde{u}_k, \tilde{v}_k)|||^2 = \frac{2(q-1)}{q} |||(u_k, v_k)|||^{q-2} Q_\lambda(\tilde{u}_k, \tilde{v}_k) + o_k(1) \to 0$$

as $k \to \infty$. This contradicts $|||(\tilde{u}_k, \tilde{v}_k)||| = 1$. Therefore, $\{(u_k, v_k)\}_k$ is bounded in $\mathbb{H}^s(\Omega)$. □

Lemma 2.9. *Then, the functional* J_λ *satisfies the* $(PS)_c$ *condition with* c *satisfying*

$$c < \hat{c} := \left(\frac{1}{2} - \frac{1}{2^*} \right) \mathcal{A}_{n,\alpha,\beta}^{\frac{2^*}{2^*-2}} + \mathcal{H}(z_*, z_*), \tag{2.23}$$

where $\mathcal{H}(z_*, z_*)$ *is defined as in Lemma 2.6.*

Proof. Let $\{(u_k, v_k)\}_k \subset \mathbb{H}^s(\Omega)$ be a $(PS)_c$ sequence for J_λ. By Lemma 2.8, the sequences $\{(u_k, v_k)\}_k$ is bounded in $\mathbb{H}^s(\Omega)$. Therefore, up to a subsequence, the following convergences hold as $k \to \infty$:

$$\begin{aligned} &(u_k, v_k) \rightharpoonup (u, v) \text{ weakly in } \mathbb{H}^s(\Omega), \\ &u_k \rightharpoonup u \quad \text{and} \quad v_k \rightharpoonup v \quad \text{ weakly in } \mathcal{X}^{1,2}(\Omega), \\ &u_k \to u \quad \text{and} \quad v_k \to v \text{ in } L^p(\Omega), \ \text{ for } p \in [1, 2^*), \\ &u_k(x) \to u(x) \quad \text{and} \quad v_k(x) \to v(x) \text{ a.e. in } \Omega, \\ &\|u_k - u\|_{L^{2^*}(\Omega)} \to l_1 \quad \text{and} \quad \|v_k - v\|_{L^{2^*}(\Omega)} \to l_2 \quad \text{for some } l_1, l_2 \ge 0, \end{aligned} \tag{2.24}$$

Additionally, by Lemma 2.5, we have

$$\nabla u_k(x) \to \nabla u(x) \quad \text{and} \quad \nabla v_k(x) \to \nabla v(x) \text{ a.e. in } \Omega \text{ as } k \to \infty.$$

Using weak convergence in Hilbert spaces and the Brézis–Lieb lemma, we obtain

$$\begin{aligned}
&\|\nabla u_k\|^2_{L^2(\Omega)} - \|\nabla u_k - \nabla u\|^2_{L^2(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)} + o_k(1),\\
&\|\nabla v_k\|^2_{L^2(\Omega)} - \|\nabla v_k - \nabla v\|^2_{L^2(\Omega)} = \|\nabla v\|^2_{L^2(\Omega)} + o_k(1),\\
&[u_k]^2_{s,\mathbb{R}^n} - [u_k - u]^2_{s,\mathbb{R}^n} = [u]^2_{s,\mathbb{R}^n} + o_k(1),\\
&[v_k]^2_{s,\mathbb{R}^n} - [v_k - v]^2_{s,\mathbb{R}^n} = [v]^2_{s,\mathbb{R}^n} + o_k(1),\\
&\|u_k\|^{2^*}_{L^{2^*}(\Omega)} - \|u_k - u\|^{2^*}_{L^{2^*}(\Omega)} = \|u\|^{2^*}_{L^{2^*}(\Omega)} + o_k(1),\\
&\|v_k\|^{2^*}_{L^{2^*}(\Omega)} - \|v_k - v\|^{2^*}_{L^{2^*}(\Omega)} = \|v\|^{2^*}_{L^{2^*}(\Omega)} + o_k(1),
\end{aligned} \tag{2.25}$$

as $k \to \infty$.

On the other hand, since the function $j(x,y) = |x|^\alpha |y|^\beta$ satisfies the hypothesis of [9, Theorem 2], we get

$$\int_\Omega \left(|u_k|^\alpha |v_k|^\beta - |u|^\alpha |v|^\beta - |u_k - u|^\alpha |v_k - v|^\beta\right) dx = o_k(1). \tag{2.26}$$

Moreover, for any $(\varphi, \phi) \in \mathbb{H}^s(\Omega)$

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{[u_k(x) - u_k(y)][\varphi(x) - \varphi(y)]}{|x-y|^{n+2s}} dydx \to \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{[u(x) - u(y)][\varphi(x) - \varphi(y)]}{|x-y|^{n+2s}} dydx \tag{2.27}$$

and

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{[v_k(x) - v_k(y)][\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dydx \to \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{[v(x) - v(y)][\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dydx \tag{2.28}$$

as $k \to \infty$.

Using that $J'_\lambda(u_k, v_k) \to 0$ as $k \to \infty$, along with (2.24), (2.27) and (2.28) we deduce

$$\begin{aligned}
o_k(1) &= \langle J'_\lambda(u_k, v_k), (u_k - u, v_k - v)\rangle\\
&= \langle u_k, u_k - u\rangle_{s,2} + \langle v_k, v_k - v\rangle_{s,2} - \frac{\lambda}{q}\int_\Omega h(x)\nabla G(u_k, v_k)\cdot(u_k - u, v_k - v)dx\\
&\quad - \int_\Omega \Big(|u_k|^{2^*-2}u_k(u_k - u) + |v_k|^{2^*-2}v_k(v_k - v)\Big)\,dx\\
&\quad - \frac{\alpha}{\alpha+\beta}\int_\Omega |u_k|^{\alpha-2}u_k|v_k|^\beta(u_k - u)dx\\
&\quad - \frac{\beta}{\alpha+\beta}\int_\Omega |u_k|^{\alpha}|v_k|^{\beta-1}v_k(v_k - v)dx\\
&= \|u_k\|^2_{s,2} - \|u\|^2_{s,2} + \|v_k\|^2_{s,2} - \|v\|^2_{s,2}\\
&\quad - \left(\int_\Omega |u_k|^{2^*}dx - \int_\Omega |u|^{2^*}dx + \int_\Omega |v_k|^{2^*}dx - \int_\Omega |v|^{2^*}dx\right)\\
&\quad - \int_\Omega \left(|u_k|^\alpha|v_k|^\beta - |u|^\alpha|v|^\beta\right)dx + o_k(1).
\end{aligned}$$

By (2.25) and (2.26), we further obtain

$$\begin{aligned}
o_k(1) &= \|u_k - u\|^2_{s,2} + \|v_k - v\|^2_{s,2} - \left(\int_\Omega |u_k - u|^{2^*}dx + \int_\Omega |v_k - v|^{2^*}dx\right)\\
&\quad - \int_\Omega |u_k - u|^\alpha |v_k - v|^\beta dx + o_k(1).
\end{aligned}$$

Now, we may assume that

$$\lim_{k\to\infty} \|(u_k - u, v_k - v)\|^2 = \lim_{k\to\infty}\int_\Omega \left(|u_k - u|^{2^*} + |v_k - v|^{2^*} + |u_k - u|^\alpha|v_k - v|^\beta\right)dx =: l \geq 0.$$

If $l=0$, the proof is completed. Assume that $l>0$, then by the definition of $\mathcal{A}_{n,\alpha,\beta}$ we have

$$\begin{aligned}\mathcal{A}_{n,\alpha,\beta} l^{\frac{2}{2^*}} &= \mathcal{A}_{n,\alpha,\beta}\left(\lim_{k\to\infty}\int_\Omega \left[|u_k-u|^{2^*}+|v_k-v|^{2^*}+|u_k-u|^\alpha|v_k-v|^\beta\right]dx\right)^{2/2^*}\\ &\le \lim_{k\to\infty}\|(u_k-u,v_k-v)\|^2=l,\end{aligned}$$

which implies

$$l\ge \mathcal{A}_{n,\alpha,\beta}^{\frac{2^*}{2^*-2}}.$$

Using (2.24), (2.25) and the definition of J_λ, we obtain

$$\begin{aligned}
&c+o_k(1)=J_\lambda(u_k,v_k)=J_\lambda(u_k,v_k)-\frac{1}{2}\langle J_\lambda'(u_k,v_k),(u_k,v_k)\rangle\\
&=-\lambda\left(\frac1q-\frac12\right)\int_\Omega h(x)G(u_k,v_k)dx\\
&+\left(\frac12-\frac1{2^*}\right)\int_\Omega\left(|u_k|^{2^*}+|v_k|^{2^*}\right)dx+\left(\frac12-\frac1{2^*}\right)\int_\Omega|u_k|^\alpha|v_k|^\beta dx\\
&\ge-\lambda\left(\frac1q-\frac12\right)\int_\Omega h(x)G(u,v)dx\\
&\quad+\left(\frac12-\frac1{2^*}\right)\int_\Omega\left(|u_k-u|^{2^*}+|u|^{2^*}+|v_k-v|^{2^*}+|v|^{2^*}\right)dx\\
&\quad+\left(\frac12-\frac1{2^*}\right)\int_\Omega\left(|u_k-u|^\alpha|v_k-v|^\beta+|u|^\alpha|v|^\beta\right)dx+o_k(1)\\
&\ge\left(\frac12-\frac1{2^*}\right)\left[\int_\Omega\left(|u_k-u|^{2^*}+|v_k-v|^{2^*}\right)dx+\int_\Omega|u_k-u|^\alpha|v_k-v|^\beta dx\right]\\
&\quad+J_\lambda(u,v)+o_k(1).
\end{aligned}$$

Thus,

$$c\ge\left(\frac12-\frac1{2^*}\right)\mathcal{A}_{n,\alpha,\beta}^{\frac{2^*}{2^*-2}}+\mathcal{H}(z_*,z_*),$$

which contradicts (2.23). Therefore $l=0$ and

$$\lim_{k\to\infty}\int_\Omega|u_k-u|^{2^*}dx=\lim_{k\to\infty}\int_\Omega|v_k-v|^{2^*}dx=\lim_{k\to\infty}\int_\Omega|u_k-u|^\alpha|v_k-v|^\beta dx=0.$$

Thus, up to a subsequence, $(u_k,v_k)\to(u,v)$ strongly in $\mathbb{H}^s(\Omega)$ as $k\to\infty$. □

3. Nehari manifold method

We now study problem (1.1) using the Nehari manifold method. To that end, we introduce the Nehari manifold

$$\mathcal{N}_\lambda := \{(u,v)\in\mathbb{H}^s(\Omega)\setminus(0,0)\colon \langle J_\lambda'(u,v),(u,v)\rangle=0\},$$

and define the auxiliary functional

$$\Psi_\lambda(u,v):=\langle J_\lambda'(u,v),(u,v)\rangle.$$

Then, for any $(u,v)\in\mathcal{N}_\lambda$ a straightforward computation shows that

$$\begin{aligned}
\langle\Psi_\lambda'(u,v),(u,v)\rangle&=2\|(u,v)\|^2-2^*\int_\Omega\left(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta\right)dx-qQ_\lambda(u,v)\\
&=(2-q)\|(u,v)\|^2-(2^*-q)\int_\Omega\left(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta\right)dx\\
&=(2-2^*)\|(u,v)\|^2-(q-2^*)Q_\lambda(u,v)\\
&=(2-2^*)\int_\Omega\left(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta\right)dx-(q-2)Q_\lambda(u,v).
\end{aligned}\tag{3.1}$$

We now split $\mathcal{N}_\lambda$ into the following three subsets:

$$\mathcal{N}_\lambda^+ \colon \{(u,v) \in \mathcal{N}_\lambda \coloneqq \langle \Psi_\lambda'(u,v),(u,v)\rangle > 0\},$$
$$\mathcal{N}_\lambda^0 \coloneqq \{(u,v) \in \mathcal{N}_\lambda \colon \langle \Psi_\lambda'(u,v),(u,v)\rangle = 0\},$$
$$\mathcal{N}_\lambda^- \coloneqq \{(u,v) \in \mathcal{N}_\lambda \colon \langle \Psi_\lambda'(u,v),(u,v)\rangle < 0\},$$

and investigate some of their key properties.

Define

$$\Lambda \coloneqq \frac{1}{\overline{G}Q_\theta}\left(\frac{2-q}{3(2^*-q)}\right)^{\frac{2-q}{2^*-2}} \mathcal{S}_n^{\frac{2^*-q}{2^*-2}}.$$

Lemma 3.1. *If $\lambda \in (0,\Lambda)$, then $\mathcal{N}^0 = \emptyset$.*

Proof. Assume by contradiction that there exists $\lambda \in (0,\Lambda)$ for which $\mathcal{N}_\lambda^0 \neq \emptyset$. For any $(u,v) \in \mathcal{N}_\lambda^0$, equation (3.1) implies

$$\|\!|(u,v)|\!\|^2 = \frac{2^*-q}{2-q}\int_\Omega \left(|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta\right) dx, \tag{3.2}$$

and also

$$\|\!|(u,v)|\!\|^2 = \frac{2^*-q}{2^*-2} Q_\lambda(u,v). \tag{3.3}$$

Since $\alpha + \beta = 2^*$, Young's inequality together with the definition of $\mathcal{S}_n$, yield

$$\begin{aligned} \int_\Omega |u|^\alpha |v|^\beta dx &\le \frac{\alpha}{2^*}\int_\Omega |u|^{2^*} dx + \frac{\beta}{2^*}\int_\Omega |v|^{2^*} dx \\ &\le \frac{\alpha}{2^*}\mathcal{S}_n^{-\frac{2^*}{2}} \|u\|_{s,2}^{2^*} + \frac{\beta}{2^*}\mathcal{S}_n^{-\frac{2^*}{2}} \|v\|_{s,2}^{2^*}. \end{aligned} \tag{3.4}$$

Combining (3.2) with (3.4) gives

$$\begin{aligned} \|\!|(u,v)|\!\|^2 &\le \frac{2^*-q}{2-q}\left[\left(1+\frac{\alpha}{2^*}\right)\mathcal{S}_n^{-\frac{2^*}{2}}\|u\|_{s,2}^{2^*} + \left(1+\frac{\beta}{2^*}\right)\mathcal{S}_n^{-\frac{2^*}{2}}\|v\|_{s,2}^{2^*}\right] \\ &\le 3\frac{2^*-q}{2-q}\mathcal{S}_n^{-\frac{2^*}{2}}\|\!|(u,v)|\!\|^{2^*}, \end{aligned}$$

so that

$$\|\!|(u,v)|\!\| \ge \left(\frac{(2-q)\mathcal{S}_n^{\frac{2^*}{2}}}{3(2^*-q)}\right)^{\frac{1}{2^*-2}}. \tag{3.5}$$

On the other hand, using (2.19) and (3.3), we deduce

$$\|\!|(u,v)|\!\| \le \left(\lambda \overline{G} Q_\theta \mathcal{S}_n^{-\frac{q}{2}}\right)^{\frac{1}{2-q}}. \tag{3.6}$$

Consequently, (3.5) and (3.6) yield

$$\lambda \ge \frac{1}{\overline{G}Q_\theta}\left(\frac{2-q}{3(2^*-q)}\right)^{\frac{2-q}{2^*-2}} \mathcal{S}_n^{\frac{2^*-q}{2^*-2}} = \Lambda,$$

a contradiction. Therefore, $\mathcal{N}_\lambda^0 = \emptyset$ for every $\lambda \in (0,\Lambda)$. □

Lemma 3.2. *The functional J_λ is coercive and bounded below on $\mathcal{N}_\lambda$*

Proof. From (2.19) one obtains

$$J_\lambda(u,v) \ge \frac{2^*-2}{22^*}\|\!|(u,v)|\!\|^2 - \frac{2^*-q}{q2^*}\lambda \overline{G} Q_\theta \mathcal{S}_n^{-\frac{q}{2}}\|\!|(u,v)|\!\|^q$$

for every $(u,v) \in \mathcal{N}_\lambda$. The affirmation follows from the fact that $q \in (1,2)$. □

Arguing as in [11, Theorem 2.3], we obtain the following result.

Lemma 3.3. *Suppose that (u_0, v_0) is a local minimizer for J_λ on $\mathcal{N}_\lambda$ and that $(u_0, v_0) \notin \mathcal{N}_\lambda^0$. Then $J'_\lambda(u_0, v_0) = 0$.*

Let
$$\Lambda_0 := \frac{q\Lambda}{2},$$
If $\lambda \in (0, \Lambda_0)$, then we have
$$\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-.$$
We also define
$$\eta_\lambda := \inf_{(u,v)\in\mathcal{N}_\lambda} J_\lambda(u, v), \quad \eta_\lambda^+ := \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u, v) \quad \text{and} \quad \eta_\lambda^- = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u, v).$$

Lemma 3.4. *If $\lambda \in (0, \Lambda_0)$ then*

(1) $\eta_\lambda \le \eta_\lambda^+ < 0$.

(2) There exists a constant $d_0 = d_0(q, N, \lambda) > 0$ such that
$$\eta_\lambda^- > d_0.$$

Proof. (1) For any $(u, v) \in \mathcal{N}_\lambda^+$, equation (3.1) implies
$$\frac{2-q}{2^*-q} |||(u, v)|||^2 > \int_\Omega \left(|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta\right) dx.$$
Therefore
$$\begin{aligned} J_\lambda(u, v) &= \left(\frac{1}{2} - \frac{1}{q}\right) |||(u, v)|||^2 - \left(\frac{1}{2^*} - \frac{1}{q}\right) \int_\Omega \left(|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta\right) dx \\ &\le \frac{2-q}{q} \left(\frac{1}{2^*} - \frac{1}{2}\right) |||(u, v)|||^2 < 0. \end{aligned}$$
Consequently, by the definition of η_λ and η_λ^+ we obtain the desired result.

(2) For $(u, v) \in \mathcal{N}_\lambda^-$ and $\lambda \in (0, \Lambda_0)$, arguing as in Lemma 3.1, we deduce
$$|||(u, v)||| \ge \left(\frac{(2-q)\mathcal{S}_n^{\frac{2^*}{2}}}{3(2^*-q)}\right)^{\frac{1}{2^*-2}}.$$
Using this together with Lemma 3.2 gives
$$\begin{aligned} J_\lambda(u, v) &\ge \frac{2^*-2}{22^*} |||(u, v)|||^2 - \frac{2^*-q}{q2^*} \lambda \overline{G} Q_\theta \mathcal{S}_n^{-\frac{q}{2}} |||(u, v)|||^q \\ &\ge \left(\frac{2^*-2}{22^*} \left(\frac{(2-q)\mathcal{S}_n^{\frac{2^*}{2}}}{3(2^*-q)}\right)^{\frac{2-q}{2^*-2}} - \frac{2^*-q}{q2^*} \lambda \overline{G} Q_\theta \mathcal{S}_n^{-\frac{q}{2}}\right) \left(\frac{(2-q)\mathcal{S}_n^{\frac{2^*}{2}}}{3(2^*-q)}\right)^{\frac{q}{2^*-2}} \\ &=: d_0 > 0. \end{aligned}$$
□

For each $(u, v) \in \mathbb{H}^s(\Omega) \setminus \{(0, 0)\}$, define
$$T := \left(\frac{(2-q)|||(u, v)|||^2}{(2^*-q)\int_\Omega \left(|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta\right) dx}\right)^{\frac{1}{2^*-2}} > 0.$$
Then, following the argument in [21, Lemma 2.7], we are able to prove the next lemma.

Lemma 3.5. *Let $\lambda \in (0, \Lambda_0)$. Then, for every $(u, v) \in \mathbb{H}^s(\Omega) \setminus \{(0, 0)\}$, we have*

(1) If $Q_\lambda(u,v) \le 0$, then there exists a unique $\tau^- > T$ such that

$$(\tau^- u, \tau^- v) \in \mathcal{N}_\lambda^- \quad \text{and} \quad J_\lambda(\tau^- u, \tau^- v) = \sup_{t\ge 0} J_\lambda(tu, tv).$$

(2) If $Q_\lambda(u,v) > 0$, then there exist unique numbers τ^+ and τ^- such that $0 < \tau^+ < T < \tau^-$, $(\tau^\pm u, \tau^\pm v) \in \mathcal{N}_\lambda^\pm$,

$$J_\lambda(\tau^+ u, \tau^+ v) = \inf_{t\in[0,T]} J_\lambda(tu, tv) \quad \text{and} \quad J_\lambda(\tau^- u, \tau^- v) = \sup_{t\ge 0} J_\lambda(tu, tv).$$

Corollary 3.6. *Let $\lambda \in (0, \Lambda_0)$. Then,*

$$\eta_\lambda^- = \inf\left\{\sup_{t\ge 0} J_\lambda(tu, tv) \colon (u,v) \in \mathbb{H}^s(\Omega) \setminus \{(0,0)\}.\right\}$$

The following result highlights the fundamental difference between our setting and the purely fractional case studied in [5].

Lemma 3.7. *Let $n \ge 3$, $q \in (1,2)$ and assume that conditions **(H1)**–**(H3)** hold. Then there exist nonnegative $(u_0, v_0) \in \mathbb{H}^s(\Omega) \setminus \{(0,0)\}$ and a constant $\Lambda^* > 0$ such that for all $\lambda \in (0, \Lambda^*)$ one has*

$$\sup_{t\ge 0} J_\lambda(tu_0, tv_0) < \hat{c} \tag{3.7}$$

where $\hat{c}$ is defined in Lemma 2.9. Moreover, one has $\eta_\lambda^- < \hat{c}$ for every $\lambda \in (0, \Lambda^*)$.

Proof. Let $\varphi_0 \in C^\infty(\mathbb{R})$ be a nonincreasing cut-off function given by

$$\varphi_0(t) := \begin{cases} 1, & \text{if } t \in \left[0, \frac{1}{2}\right], \\ 0, & \text{if } t \ge 1. \end{cases}$$

Without loss of generality, we assume that $0 \in \Omega$. For $\rho > 0$ sufficiently small (so that $\overline{B(0,\rho)} \subset \Omega$, define

$$\varphi_\rho(x) \coloneqq \varphi_0\left(\frac{|x|}{\rho}\right).$$

For each $\epsilon > 0$, set

$$\mathcal{U}_\epsilon(x) \coloneqq \frac{\epsilon^{\frac{n-2}{2}}}{(|x|^2 + \epsilon^2)^{\frac{n-2}{2}}} \quad \text{and} \quad w_\epsilon(x) \coloneqq \frac{\varphi_\rho(x)\mathcal{U}_\epsilon(x)}{\|\varphi_\rho \mathcal{U}_\epsilon\|_{L^{2^*}}}.$$

Let

$$2^* > p > \begin{cases} 2 + \dfrac{4s}{n-2} & \text{if } n \ge 4 \\ 4 & \text{if } n = 3. \end{cases} \tag{3.8}$$

Then for each $q \in (1,2)$, using that $|w_\epsilon| \le 1$, one may show that

$$\int_{\mathbb{R}^n} w_\epsilon(x)^q dx \ge \int_{\mathbb{R}^n} w_\epsilon(x)^p dx \ge C_0 \epsilon^{n - p\frac{n-2}{2}} \tag{3.9}$$

with $C_0 = C_0(n,p) > 0$, see, e.g. [7, Lemma 4.10]. Also

$$\int_\Omega |\nabla w_\epsilon(x)|^2 dx = \mathcal{S}_n + O(\epsilon^{n-2})$$

and

$$\iint_{\mathbb{R}^{2n}} \frac{|w_\epsilon(x) - w_\epsilon(y)|^2}{|x-y|^{n+2s}} dy dx = O(\epsilon^{2-2s})$$

as $\epsilon \to 0^+$.

Next, define the functional $\mathcal{J} \colon \mathbb{H}^s(\Omega) \to \mathbb{R}$ by

$$\mathcal{J}(u,v) \coloneqq \frac{1}{2}\|(u,v)\|^2 - \frac{1}{2^*}\int_\Omega \left(|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta\right) dx.$$

Then

$$\begin{aligned}\mathcal{J}(tw_\epsilon, tt_*w_\epsilon) &= \frac{1}{2}\|\!|(tw_\epsilon, tt_*w_\epsilon)|\!\|^2 - \frac{1}{2^*}\int_\Omega \left(|tw_\epsilon|^{2^*} + |tt_*w_\epsilon|^{2^*} + |tw_\epsilon|^\alpha |tt_*w_\epsilon|^\beta\right)dx \\ &= \frac{t^2}{2}(1+t_*^2)\|w_\epsilon\|_{s,2}^2 - \frac{t^{2^*}}{2^*}\left(1 + t_*^\beta + t_*^{2^*}\right)\int_\Omega |w_\epsilon|^{2^*}dx,\end{aligned}$$

where t_* is a root of (2.5). A standard calculation shows that

$$\sup_{t\geq 0}\left(\frac{t^2}{2}A_\epsilon - \frac{t^{2^*}}{2^*}B_\epsilon\right) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\frac{A_\epsilon}{B_\epsilon^{2/2^*}}\right)^{\frac{2^*}{2^*-2}},$$

with

$$A_\varepsilon \coloneqq (1+t_*^2)\|w_\epsilon\|_{s,2}^2 \quad \text{and} \quad B_\varepsilon \coloneqq \left(1 + t_*^\beta + t_*^{2^*}\right)\int_\Omega |w_\epsilon|^{2^*}dx.$$

Thus, taking $(u_0, v_0) = (w_\epsilon, t_*w_\epsilon)$ and $t_\varepsilon = \left(\dfrac{A_\varepsilon}{B_\varepsilon}\right)^{\frac{1}{2^*-2}}$, one obtains

$$\begin{aligned}\sup_{t\geq 0}\mathcal{J}(tu_0, tv_0) &= \mathcal{J}(t_\epsilon u_0, t_\epsilon v_0) \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\frac{(1+t_*^2)\|w_\epsilon\|_{s,2}^2}{\left(\left(1 + t_*^\beta + t_*^{2^*}\right)\int_\Omega |w_\epsilon|^{2^*}\right)^{2/2^*}}\right)^{\frac{2^*}{2^*-2}} \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\left[\frac{1+t_*^2}{\left(1 + t_*^\beta + t_*^{2^*}\right)^{2/2^*}}\frac{\|w_\epsilon\|_{s,2}^2}{\left(\int_\Omega |w_\epsilon|^{2^*}dx\right)^{2/2^*}}\right]^{\frac{2^*}{2^*-2}} \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\mathfrak{f}(t_*)^{\frac{2^*}{2^*-2}}\left(\mathcal{S}_n + O(\epsilon^{n-2}) + O(\epsilon^{2-2s})\right)^{\frac{2^*}{2^*-2}} \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\mathfrak{f}(t_*)^{\frac{2^*}{2^*-2}}\left(\mathcal{S}_n + O(\epsilon^{\kappa_{s,n}})\right)^{\frac{2^*}{2^*-2}} \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\mathfrak{f}(t_*)\mathcal{S}_n\right)^{\frac{2^*}{2^*-2}} + O(\epsilon^{\kappa_{s,n}})\end{aligned}$$

where

$$\kappa_{s,n} \coloneqq \begin{cases} 1 & \text{if } n = 3, \\ 2-2s & \text{if } n > 4. \end{cases} \tag{3.10}$$

On the other hand, since

$$\mathcal{H}(z_*, z_*) = \frac{(2^*-q)^{\frac{2}{2-q}}}{2^*(2^*-2)^{\frac{q}{2-q}}}\left(1 - \frac{2}{q}\right)\left(\lambda\overline{G}Q_\theta\right)^{\frac{2}{2-q}}\mathcal{S}_n^{-\frac{q}{2-q}},$$

one may choose Λ^* small enough so that for every $\lambda \in (0, \Lambda^*)$

$$\left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\mathfrak{f}(t_*)\mathcal{S}_n\right)^{\frac{2^*}{2^*-2}} + \mathcal{H}(z_*, z_*) > 0.$$

Consequently, by the continuity of J_λ in $\mathbb{H}^s(\Omega)$ and $J_\lambda(0,0) = 0$, there is $t_0 \in (0,1)$ such that, for every $\lambda \in (0, \Lambda^*)$

$$\sup_{t\in[0,t_0]}\mathcal{J}(tu_0, tv_0) \leq \frac{t^2}{2}(1+t_*^2)\|w_\epsilon\|_{s,2}^2 < \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\mathfrak{f}(t_*)\mathcal{S}_n\right)^{\frac{2^*}{2^*-2}} + \mathcal{H}(z_*, z_*). \tag{3.11}$$

Furthermore, combining **(H3)** with (3.9) and setting

$$\underline{G} := \frac{G(1, t_*)}{1 + t_*^q} > 0,$$

we get

$$\begin{aligned}\sup_{t\geq t_0} J_\lambda(tu_0, tv_0) &\leq \sup_{t\geq t_0}\left[\mathcal{J}(tw_\epsilon, tt_*w_\epsilon) - \frac{\lambda}{q}t^q\int_\Omega h(x)G(w_\epsilon, t_*w_\epsilon)dx\right]\\ &\leq \sup_{t\geq t_0}\mathcal{J}(tu_0, tv_0) - \frac{\lambda\underline{G}}{q}t_0^q\int_\Omega h(x)\left(|w_\epsilon|^q + |t_*w_\epsilon|^q\right)dx\\ &\leq \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\mathfrak{f}(t_*)\mathcal{S}_n\right)^{\frac{2^*}{2^*-2}} - \lambda\frac{\underline{G}t_0^q}{q}\left(1 + t_*^q\right)\int_\Omega h(x)|w_\epsilon(x)|^q dx\\ &\leq \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\mathfrak{f}(t_*)\mathcal{S}_n\right)^{\frac{2^*}{2^*-2}} + C\epsilon^{\kappa_{s,n}} - \lambda\widehat{C}\epsilon^{n-p\frac{n-2}{2}},\end{aligned}$$

here

$$\widehat{C} = \frac{\underline{G}t_0^q}{q}\left(1 + t_*^q\right)\hat{h}C_0.$$

On the other hand, by (3.8) and (3.10) we have $\kappa_{s,n} > n - p\frac{n-2}{2}$. Then, taking ϵ small enough such that

$$C\epsilon^{\kappa_{s,n}} - \lambda\widehat{C}\epsilon^{n-p\frac{n-2}{2}} < \mathcal{H}(z_*, z_*), \quad \forall\lambda \in (0, \Lambda^*),$$

for every $\lambda \in (0, \Lambda^*)$ we have

$$\sup_{t\geq t_0} J_\lambda(tu_0, tv_0) \leq \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\mathfrak{f}(t_*)\mathcal{S}_n\right)^{\frac{2^*}{2^*-2}} + \mathcal{H}(z_*, z_*). \tag{3.12}$$

From (2.6) combining with (3.11) and (3.12) we get (3.7).

Finally, we prove that $\eta_\lambda^- < \hat{c}$ for all $\lambda \in (0, \lambda_0)$. Recall that

$$(u_0, v_0) = (w_\epsilon, t^*w_\epsilon).$$

By Lemma 3.5 there is $\tau^- > 0$ such that $(\tau^-u_0, \tau^-v_0) \in \mathcal{N}_\lambda^-$. By the definition of η_λ^-, we conclude

$$\eta_\lambda^- \leq J_\lambda(\tau^-u_0, \tau^-v_0) \leq \sup_{t\geq 0} J_\lambda(tu_0, tv_0) < \hat{c}, \quad \forall\lambda \in (0, \Lambda^*).$$

□

4. Proof of the main result

In this section we provide the full proof of Theorem 1.1.

Proof of Theorem 1.1. We begin by noting that, by (2.9), Hölder inequality and (2.2), we obtian

$$\begin{aligned}\lambda\int_\Omega h(x)G(u, v)dx &\leq \lambda\overline{G}\int_\Omega h(x)\left(|u|^q + |v|^q\right)dx\\ &\leq \lambda\overline{G}Q_\theta\left(\|u\|_{L^{2^*}(\Omega)}^q + \|v\|_{L^{2^*}(\Omega)}^q\right)\\ &\leq \lambda\overline{G}Q_\theta\mathcal{S}_n^{-\frac{q}{2}}\|(u, v)\|^q.\end{aligned} \tag{4.1}$$

Next, applying (2.2), (2.5) and (4.1), we obtain

$$J_\lambda(u, v) \geq f(\rho) - g_\lambda(\rho),$$

where

$$\rho := \|(u, v)\|, \quad f(\rho) := \frac{1}{2}\rho^2 - \left(\frac{\mathcal{S}_n^{-\frac{2^*}{2}}}{2^*} + \frac{\mathcal{A}_{n,\alpha,\beta}^{-\frac{2^*}{2}}}{2^*}\right)\rho^{2^*} \quad \text{and} \quad g_\lambda(\rho) := \lambda\overline{G}Q_\theta\mathcal{S}_n^{-\frac{q}{2}}\rho^q.$$

Since $2 < 2^*$, the function f attains its maximum at

$$\rho_0 := \left(\mathcal{S}_n^{-\frac{2^*}{2}} + \mathcal{A}_{n,\alpha,\beta}^{-\frac{2^*}{2}} \right)^{\frac{-1}{2^*-2}}.$$

Furthermore, $f(\rho_0) > 0$, so there exists $\mu > 0$ such that

$$\inf \{J_\lambda(u,v) \colon (u,v) \in \mathbb{H}^s(\Omega), |||(u,v)||| = \rho_0\} \geq f(\rho_0) - g_\lambda(\rho_0) > 0,$$

for any $\lambda \in (0, \mu)$.

Now, define the closed ball

$$B_0 := \{(u,v) \in \mathbb{H}^s(\Omega) \colon |||(u,v)||| \leq \rho_0\}.$$

For any $(u,v) \in \mathbb{H}^s(\Omega) \setminus \{(0,0)\}$ there exists a sufficiently small $t > 0$ such that $(tu, tv) \in B_0$ and

$$\begin{aligned} J_\lambda(tu,tv) &= \frac{t^2}{2}|||(u,v)|||^2 - \frac{t^q}{q} Q_\lambda(u,v) - \frac{t^{2^*}}{2^*}\left(\int_\Omega \left(|u|^{2^*} + |v|^{2^*} \right) dx + \int_\Omega |u|^\alpha |v|^\beta dx \right) \\ &< 0, \end{aligned}$$

where the inequality follows from $1 < q < 2 < 2^*$. Hence, we conclude that

$$-\infty < c := \inf \{J_\lambda(u,v) \colon (u,v) \in B_0\} < 0. \tag{4.2}$$

Applying Ekeland's variational principle, there exists a sequence $\{(u_k, v_k)\}_{k\in\mathbb{N}} \subset B_0$ such that

$$J_\lambda(u_k, v_k) \leq c + \frac{1}{k}, \tag{4.3}$$

and for all $(u,v) \in B_0$,

$$J_\lambda(u_k, v_k) \leq J_\lambda(u_k, v_k) + \frac{1}{k}|||(u_k - u, v_k - v)|||. \tag{4.4}$$

Our next goal is to show that $\{(u_k, v_k)\}$ is a $(PS)_c$ sequence for J_λ. By (4.3), we have

$$\lim_{k\to\infty} J_\lambda(u_k, v_k) = c. \tag{4.5}$$

On the other hand, by (4.4), (u_k, v_k) is the minimizer of

$$J^*(u,v) := J_\lambda(u,v) + \frac{1}{k}|||(u_k - u, v_k - v)|||$$

on B_0. Moreover, by (4.2), (4.3), and (4.4), there is $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ we get

$$(u_k, v_k) \in B_\varepsilon := \{(u,v) \in \mathbb{H}^s(\Omega) \colon |||(u,v)||| \leq \rho_0 - \varepsilon\}.$$

Thus, for any $(\phi, \varphi) \in \mathbb{H}^s(\Omega)$ and $k \geq k_0$, we choose $t > 0$ small enough such that $(u_k + tu, v_k + tv) \in B_0$ and

$$\begin{aligned} 0 &\leq \frac{J^*(u_k + t\phi, v_k + t\varphi) - J^*(u_k, v_k)}{t} \\ &= \frac{J_\lambda(u_k + t\phi, v_k + t\varphi) - J_\lambda(u_k, v_k)}{t} + \frac{1}{k}|||(\phi,\varphi)|||. \end{aligned}$$

Then, for any $(\phi, \varphi) \in \mathbb{H}^s(\Omega)$ and $k \geq k_0$,

$$\langle J_\lambda'(u_k, v_k), (\phi, \varphi)\rangle = \lim_{t\to 0^+} \frac{J_\lambda(u_k + t\phi, v_k + t\varphi) - J_\lambda(u_k, v_k)}{t} \geq -\frac{1}{k}|||(\phi,\varphi)|||.$$

Therefore

$$\|J_\lambda'(u_k, v_k)\|_{\mathbb{H}^s(\Omega)^*} \leq \frac{1}{k} \quad \forall k \geq k_0,$$

where $\mathbb{H}^s(\Omega)^*$ denotes the dual space of $\mathbb{H}^s(\Omega)$. Hence

$$\lim_{k\to\infty} J_\lambda'(u_k, v_k) = 0. \tag{4.6}$$

Thus, by (4.5) and (4.6), we conclude that $\{(u_k, v_k)\}$ is a $(PS)_c$ sequence for J_λ.

Further, there is $\bar{\Lambda} \in (0, \mu)$ such that for any $\lambda \in (0, \bar{\Lambda})$ we have $c < \hat{c}$, where $\hat{c}$ is defined in Lemma 2.9. Applying Lemma 2.9, we obtain that for each $\lambda \in (0, \Lambda^*)$ there exists $(u, v) \in B_\varepsilon$ such that

$$(u_k, v_k) \to (u, v) \quad \text{strongly in } \mathbb{H}^s(\Omega).$$

Consequently, $J_\lambda(u, v) = c < 0$ and (u, v) is a nontrivial solution of (1.1).

To complete the proof, observe that:

- $(|u|, |v|) \in B_\varepsilon$;
- $J_\lambda(|u|, |v|) = J_\lambda(u, v) = c$;
- $J'_\lambda(|u|, |v|) = 0$.

Therefore $(|u|, |v|)$ is a nontrivial nonnegative solution of (1.1). By the strong maximum principle (see [24, Theorem 1.2 and Remark 1.3]), $(|u|, |v|)$ is positive in Ω.

Our next goal is to establish the existence of a second positive solution to (1.1). Set

$$d \coloneqq \inf \left\{ \sup \left\{ J_\lambda(\gamma(r)) \colon r \in [0, 1] \right\} : \gamma \in \Gamma \right\}$$

where

$$\Gamma \coloneqq \left\{ \gamma \in C^0([0, 1], \mathbb{H}^s(\Omega)) \colon \gamma(0) = (0, 0), J_\lambda(\gamma(1)) < 0 \right\}.$$

Observe that $c \le 0 \le d$.

On the other hand if $(u, v) \in \mathbb{H}^s(\Omega) \setminus \{(0, 0)\}$, then

$$\begin{aligned}
&\lim_{t\to\infty} J_\lambda(tu, tv) = \\
&= \lim_{t\to\infty} \frac{t^2}{2} |||(u, v)|||^2 - \frac{t^q}{q} Q_\lambda(u, v) - \frac{t^{2^*}}{2^*} \left(\int_\Omega \left(|u|^{2^*} + |v|^{2^*} \right) dx + \int_\Omega |u|^\alpha |v|^\beta dx \right) \\
&= -\infty,
\end{aligned}$$

due to $1 < q < 2 < 2^*$. Thus, for any $(u, v) \in \mathbb{H}^s(\Omega) \setminus \{(0, 0)\}$, there exists $a > 0$ such that $J_\lambda(au, av_0) < 0$. Therefore

$$d \le \sup_{t\in[0,1]} J_\lambda(tau, tav) \le \sup_{t\ge 0} J_\lambda(tu, tv) \quad \forall (u, v) \in \mathbb{H}^s(\Omega) \setminus \{(0, 0)\}.$$

Then, by Corollary 3.6, we have that

$$d \le \eta_\lambda^-. \tag{4.7}$$

Now define $\lambda_0 \coloneqq \min\{\bar{\Lambda}, \Lambda^*\}$, where Λ^* is given in Lemma 3.7. Then, by Lemma 3.7 and (4.7),

$$d \le \eta_\lambda^- < \hat{c}. \tag{4.8}$$

By the mountain pass lemma (see [3]), there exists a sequence $\{(u_k, v_k)\}_{k\in\mathbb{N}} \subset \mathbb{H}^s$ such that

$$\lim_{k\to\infty} J_\lambda(u_k, v_k) = d \quad \text{and} \quad \lim_{k\to\infty} J'_\lambda(u_k, v_k) = 0.$$

By (4.8) and Lemma 2.9, up to a subsequence, the sequence $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ converges strongly in $\mathbb{H}^s(\Omega)$ to some (z, w). Consequently, we have

$$J_\lambda(z, w) = d \text{ and } J'_\lambda(z, w) = 0.$$

Thus (z, w) is a second nontrivial solution of (1.1).

Claim: $(z, w) \in \mathcal{N}_\lambda^-$.

Suppose, by contradiction, that $(z, w) \in \mathcal{N}_\lambda^+$. Then, by equation (3.1), we have

$$\frac{2 - q}{2^* - q} |||(z, w)|||^2 > \int_\Omega \left(|z|^{2^*} + |w|^{2^*} + |z|^\alpha |w|^\beta \right) dx.$$

Therefore

$$\begin{aligned}
J_\lambda(z, w) &= \left(\frac{1}{2} - \frac{1}{q} \right) |||(z, w)|||^2 - \left(\frac{1}{2^*} - \frac{1}{q} \right) \int_\Omega \left(|z|^{2^*} + |w|^{2^*} + |z|^\alpha |w|^\beta \right) dx \\
&\le \frac{2 - q}{q} \left(\frac{1}{2^*} - \frac{1}{2} \right) |||(z, w)|||^2 < 0.
\end{aligned}$$

This leads to a contradiction, since $J_\lambda(z,w) = d > 0$. Thus, $(z,w) \in \mathcal{N}_\lambda^-$, as claimed.

On the other hand by the definition of η_λ^- and (4.7), we conclude that $d = \eta_\lambda^-$.

Finally, since $J_\lambda(z,w) = J_\lambda(|z|,|w|)$ and $(|z|,|w|) \in \mathcal{N}_\lambda^-$, we may assume that (z,w) is a nontrivial nonnegative solution of (1.1). By the strong maximum principle (see [24, Theorem 1.2 and Remark 1.3]), (z,w) is positive in Ω. □

5. Declarations

Ethical Approval: Not applicable

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