

EXISTENCE OF POSITIVE SOLUTIONS FOR LOCAL-NONLOCAL ELLIPTIC PROBLEMS IN EXTERIOR DOMAINS

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ABSTRACT. In this paper, we investigate the existence of positive solutions for a class of elliptic problems driven by a mixed local-nonlocal operator in exterior domains. Specifically, we consider the problem governed by the Laplacian and the fractional Laplacian, focusing on the critical Sobolev exponent range and the interplay between local and nonlocal terms. Our approach, inspired by variational methods and the foundational work of Benci and Cerami (see [6]), extends classical existence results to a setting where the exterior geometry and fractional effects play a crucial role.

CONTENTS

1. Introduction and main results	1
2. Function space	3
3. Associated functionals	4
Problem in exterior domain	4
Problem in $\mathbb{R}^N$ with $\varepsilon > 0$	4
Problem in $\mathbb{R}^N$ with $\varepsilon = 0$	5
4. A nonexistence of ground state solution result	5
5. Technical results	7
6. Proof of Theorem 1.1	15
7. Declarations	17
References	17

1. INTRODUCTION AND MAIN RESULTS

In the seminal paper [6], Benci and Cerami studied the nonlinear elliptic problem

$$(P) \quad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega \text{ and } u \not\equiv 0, \\ u = 0 & \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded domain with non-empty smooth boundary, $N \geq 3$, and $2 \leq p < 2^* := \frac{2N}{N-2}$. They proved that for any fixed point $x_0 \in \mathbb{R}^N \setminus \Omega$ and for any $\lambda \in \mathbb{R}$, there exists $\rho = \rho(\lambda) > 0$ such that if $\mathbb{R}^N \setminus \Omega \subset B_\rho(x_0)$, then the problem (P) admits at least one positive solution.

This remarkable result sparked further research on similar classes of problems. For instance, multiplicity results for this type of problem can be found in [14–16, 34], while the existence of nodal solutions has been studied in [17]. Fractional Laplacian versions, involving purely

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nonlocal operators, were considered in [3, 4, 19], and fractional models with Choquard-type nonlinearities were investigated in [39]. The scope of this research has since expanded to other operators. For example, [2] established existence results for problems involving the diamagnetic operator and complex-valued solutions.

Motivated by these developments and inspired by the approach [6], in this paper we study the existence of solutions for the following class of local-nonlocal elliptic problems:

$$(P_\varepsilon) \quad \begin{cases} -\Delta u + \varepsilon(-\Delta)^s u + u = |u|^{p-2}u & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega \text{ and } u \not\equiv 0, \\ u = 0 & \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an exterior domain with non-empty smooth boundary $\partial\Omega$, $N \geq 3$, $s \in (0, 1)$, $p \in (2, 2^*)$ and $\varepsilon \in (0, 1)$.

We recall that the fractional Laplacian $(-\Delta)^s$ is a nonlocal operator formally defined as

$$(-\Delta)^s u(x) := C(N, s) \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where P.V. denotes the principal value of the integral, and $C(N, s)$ is a normalization constant. For convenience, we will omit this constant $C(N, s)$, as it does not affect the variational structure or the qualitative behavior of solutions.

Our main contribution is the following theorem, which extends classical variational results to a setting that combines local and nonlocal operators in exterior domains:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an exterior domain with non-empty smooth boundary $\partial\Omega$, $s \in (0, 1)$, $N > 2s$, $p \in (2, 2^*)$ and $\varepsilon \in (0, 1)$. Then there exist $\rho > 0$ and $\varepsilon_* > 0$ such that $\mathbb{R}^N \setminus \Omega \subset B(0, \rho)$ and $\varepsilon \in (0, \varepsilon_*)$, the problem (P_ε) has at least one positive solution.*

This theorem emphasizes the interplay between the classical local problem ($\varepsilon = 0$) and a mixed local-nonlocal model ($\varepsilon > 0$), which combines standard and fractional diffusion effects. Moreover, it highlights the influence of the exterior domain's geometry on the existence of solutions, a phenomenon absent in problems posed on the whole space.

Our approach combines variational methods, concentration-compactness arguments, and refined analysis of the mixed local-nonlocal operator $-\Delta + \varepsilon(-\Delta)^s$, with careful consideration of the fractional order s and the parameter ε .

The study of mixed local-nonlocal operators is motivated by their occurrence in various models in mathematical physics, ecology, and materials science. The classical Laplacian $-\Delta$ describes standard diffusion or local interactions (see, for instance, [5, 27]), while the fractional Laplacian $(-\Delta)^s$ models anomalous diffusion and long-range interactions, as observed in media with heterogeneities, memory effects, or jump processes (see, for example, [32, 33, 40, 42]). The combined operator $-\Delta + \varepsilon(-\Delta)^s$ offers a comprehensive framework that captures both local and nonlocal diffusion phenomena, which is essential for accurately modeling systems where both types of processes are relevant.

In recent years, the study of problems driven by mixed operators has garnered significant interest, not only due to the mathematical intricacies arising from the combination of two distinct operators but also because of their diverse applications. For instance, in [26] examines the operator $-\Delta + (-\Delta)^s$ to model the dynamics of a biological species where individuals disperse either via random walks or through jump processes, with the diffusion mode determined by given probabilities. Further applications of mixed operators in biological contexts are presented in [12, 24, 35, 36].

In addition, a broad range of studies has been conducted on mixed local-nonlocal operators, approaching them from multiple perspectives such as regularity theory, existence and nonexistence of solutions, variational methods, shape optimization, and eigenvalue problems. For detailed insights, see, for example, [7–11, 18, 23, 29, 38] and the references therein. Collectively, these works highlight the intricate mathematical framework and the broad relevance of mixed operators, paving the way for continued advancements in the field.

The structure of this paper is as follows. In Section 2, we introduce the functional setting and recall essential embedding results. Section 3 is devoted to the variational framework and the associated energy functionals. In Section 4, we establish a nonexistence result for ground state solutions. Section 5 contains technical lemmas that are crucial for our main proof. The proof of Theorem 1.1 is presented in Section 6.

2. FUNCTION SPACE

We now introduce a suitable functional framework for the study of mixed local-nonlocal operators.

The proof of the following result can be found, for instance, in [21].

Lemma 2.1. *If $\Omega = \mathbb{R}^N$ or if Ω has a bounded Lipschitz boundary, when there exists a continuous embedding $H^1(\Omega) \hookrightarrow H^s(\Omega)$. That is, there exists a constant $C = C(N, s) > 0$ such that*

$$[u]_{s,\Omega}^2 := \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy dx \leq C \|u\|_{H^1(\Omega)}^2, \quad \forall u \in H^1(\Omega).$$

Let $\Omega \subset \mathbb{R}^N$ be an exterior domain with non-empty smooth boundary, and let $s \in (0, 1)$ with $N > 2s$. We define the space $\mathcal{X}^{1,2}(\Omega)$ as the space $H_0^1(\Omega)$ equipped with the norm

$$\|u\|_{s,2} := \left(\|u\|_{H^1(\Omega)}^2 + [u]_{s,\mathbb{R}^N}^2 \right)^{\frac{1}{2}},$$

which is induced by the inner product

$$\langle u, v \rangle_{s,2} := \int_{\Omega} uv dx + \int_{\Omega} \nabla u \nabla v dx + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{n+2s}} dy dx.$$

With this structure, $\mathcal{X}^{1,2}(\Omega)$ becomes a Hilbert space. Moreover, according to [7, Remark 2.2], the following continuous embedding holds:

$$\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$$

where $2^* := \frac{2N}{N-2}$ denotes the critical Sobolev exponent. In particular, there exists a constant $\mathcal{S} > 0$ such that

$$\mathcal{S} \|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq \|u\|_{s,2}^2.$$

As a consequence, we have the continuous embeddings

$$\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^p(\Omega),$$

for all $p \in [2, 2^*]$ holds.

3. ASSOCIATED FUNCTIONALS

Problem in exterior domain. We denote by $\mathcal{J}_\varepsilon: \mathcal{X}^{1,2}(\Omega) \rightarrow \mathbb{R}$ the energy functional associated with problem (P_ε) , defined as

$$\mathcal{J}_\varepsilon(u) := \frac{1}{2} \left(\varepsilon \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy dx + \int_{\Omega} (|\nabla u|^2 + |u|^2) dx \right) - \frac{1}{p} \int_{\Omega} |u|^p dx.$$

Using standard arguments, one can show that $\mathcal{J}_\varepsilon \in C^1(H_0^1(\Omega), \mathbb{R})$ with Fréchet derivative given by

$$\mathcal{J}'_\varepsilon(u)\varphi = \varepsilon \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[u(x) - u(y)][\varphi(x) - \varphi(y)]}{|x - y|^{N+2s}} dy dx + \int_{\Omega} (\nabla u \nabla \varphi + u\varphi) dx - \int_{\Omega} |u|^{p-2} u \varphi dx,$$

for all $u, \varphi \in \mathcal{X}^{1,2}(\Omega)$.

The Nehari manifold associated with $\mathcal{J}_\varepsilon$ is defined by

$$\mathcal{N}_\varepsilon := \{u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\} : \mathcal{J}'_\varepsilon(u)u = 0\}$$

and the corresponding ground state energy level is given by

$$\mu_\varepsilon := \inf_{u \in \mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u).$$

A sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{X}^{1,2}(\Omega)$ is called a Palais-Smale sequence at level $c \in \mathbb{R}$ (or a $(PS)_c$ sequence) for the functional $\mathcal{J}_\varepsilon$ if

$$\mathcal{J}_\varepsilon(u_k) \rightarrow c \quad \text{and} \quad \mathcal{J}'_\varepsilon(u_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Problem in $\mathbb{R}^N$ with $\varepsilon > 0$. To prove our results, we will follow the methods used in [6]. However, to the best of our knowledge, the uniqueness of the solution for the limit problem

$$(P_{\varepsilon,\infty}) \quad \begin{cases} -\Delta u + \varepsilon(-\Delta)^s u + u = |u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

is not guaranteed.

Here $H^1(\mathbb{R}^N)$ is endowed with the norm

$$\|u\|_\varepsilon := \left(\int_{\mathbb{R}^N} (|u|^2 + |\nabla u|^2) dx + \varepsilon \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy dx \right)^{1/2}.$$

which is equivalent to the standard norm.

The associated energy functional $\mathcal{J}_{\varepsilon,\infty}: H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ is given by

$$\mathcal{J}_{\varepsilon,\infty}(u) := \frac{1}{2} \left(\varepsilon \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy dx + \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx \right) - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx.$$

Clearly, $\mathcal{J}_{\varepsilon,\infty} \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$. The Nehari manifold associated with $\mathcal{J}_{\varepsilon,\infty}$ is given by

$$\mathcal{N}_{\varepsilon,\infty} := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \mathcal{J}'_{\varepsilon,\infty}(u)u = 0\}.$$

By following the same lines of the proof of [25, Theorems 1.1 and 1.5], we can establish the following theorem.

Theorem 3.1. *For any $\varepsilon > 0$, problem $(P_{\varepsilon,\infty})$ admits a positive, radially symmetric, classical solution w_ε such that*

$$\mu_{\varepsilon,\infty} := \inf_{u \in \mathcal{N}_{\varepsilon,\infty}} \mathcal{J}_{\varepsilon,\infty}(u) = \mathcal{J}_{\varepsilon,\infty}(w_\varepsilon).$$

Furthermore, there exist constants $0 < C_1 \leq C_2$ such that, for all $|x| \geq 1$,

$$\frac{C_1}{|x|^{N+2s}} \leq w_\varepsilon(x) \leq \frac{C_2}{|x|^{N+2s}}.$$

Problem in $\mathbb{R}^N$ with $\varepsilon = 0$. To prove our main result, it is crucial to understand the behavior of certain energy levels of the functional $\mathcal{J}_\infty: H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ associated with the problem

$$(P_\infty) \quad \begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

which is defined as

$$\mathcal{J}_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx.$$

The Nehari manifold associated with $\mathcal{J}_\infty$ is

$$\mathcal{N}_\infty := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \mathcal{J}'_\infty(u)u = 0\}.$$

For $p \in (2, 2^*)$, this problem has a unique positive solution (up to translation), which we denote by ψ . This solution is smooth, radially symmetric about the origin, and monotonically decreasing. Furthermore, we have

$$\mu_\infty := \inf_{u \in \mathcal{N}_\infty} \mathcal{J}_\infty(u) = \mathcal{J}_\infty(\psi).$$

For further details, we refer the reader to [28, 41].

On the other hand, let $\mathcal{I}: H_0^1(\Omega) \rightarrow \mathbb{R}$ be the restriction of $\mathcal{J}_\infty$ to $H_0^1(\Omega)$, defined by

$$\mathcal{I}(u) := \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + |u|^2) dx - \frac{1}{p} \int_{\Omega} |u|^p dx.$$

The associated Nehari manifold is

$$\mathcal{N} := \{u \in H_0^1(\Omega) \setminus \{0\} : \mathcal{I}'(u)u = 0\},$$

and the corresponding critical level is

$$\mu := \inf_{u \in \mathcal{N}} \mathcal{I}(u).$$

4. A NONEXISTENCE OF GROUND STATE SOLUTION RESULT

In this section, we prove that problem (P_ε) does not admit a ground state solution. We begin with the following key result.

Lemma 4.1. *The levels μ_ε and $\mu_{\varepsilon, \infty}$ satisfy*

$$\mu_\varepsilon = \mu_{\varepsilon, \infty}.$$

Proof. We first observe that $\mu_{\varepsilon, \infty} \leq \mu_\varepsilon$. In fact, for any $u \in \mathcal{N}_\varepsilon$, there is $t_u > 0$ satisfying $t_u u \in \mathcal{N}_{\varepsilon, \infty}$. Hence

$$\begin{aligned} \mu_{\varepsilon, \infty} &\leq \mathcal{J}_{\varepsilon, \infty}(t_u u) \\ &= \frac{1}{2} \left(\varepsilon \iint_{\mathbb{R}^{2N}} \frac{|(t_u u)(x) - (t_u u)(y)|^2}{|x - y|^{N+2s}} dy dx + \int_{\mathbb{R}^N} (|\nabla(t_u u)|^2 + |t_u u|^2) dx \right) - \frac{1}{p} \int_{\mathbb{R}^N} |t_u u|^p dx \\ &= \frac{1}{2} \left(\varepsilon \iint_{\mathbb{R}^{2N}} \frac{|(t_u u)(x) - (t_u u)(y)|^2}{|x - y|^{N+2s}} dy dx + \int_{\Omega} (|\nabla(t_u u)|^2 + |t_u u|^2) dx \right) - \frac{1}{p} \int_{\Omega} |t_u u|^p dx \\ &= \mathcal{J}_\varepsilon(t_u u) \leq \mathcal{J}_\varepsilon(u). \end{aligned}$$

This gives the desired result.

To conclude the proof, we must show the reverse inequality.

Let $\zeta \in C^\infty(\mathbb{R}^N)$ be a smooth cutoff function such that

$$0 \leq \zeta \leq 1, \quad \zeta(x) = 0, \quad x \in B(0, \rho) \quad \text{and} \quad \zeta(x) = 1, \quad x \in \mathbb{R}^N \setminus B(0, 2\rho),$$

where ρ is chosen so that $\mathbb{R}^N \setminus \Omega \subset B(0, \rho)$.

Consider a sequence $\{y_k\} \subset \Omega$ such that $|y_k| \rightarrow +\infty$. For each $k \in \mathbb{N}$, define

$$\varphi_k(x) := \zeta(x)w_\varepsilon(x - y_k),$$

where $w_\varepsilon \in \mathcal{N}_{\varepsilon, \infty}$ is the ground state solution provided by Theorem 3.1.

Let $t_k > 0$ such that $t_k \varphi_k \in \mathcal{N}_\varepsilon$. Then,

$$(4.1) \quad \mu_\varepsilon \leq \mathcal{J}_\varepsilon(t_k \varphi_k) = \mathcal{J}_{\varepsilon, \infty}(t_k \varphi_k), \quad \forall k \in \mathbb{N}.$$

Note that for any $x, y, z \in \mathbb{R}^N$, there is $k_0 = k_0(x, y, z) \in \mathbb{N}$ such that for all $k \geq k_0$

$$|\zeta(z + y_k)w_\varepsilon(z) - w_\varepsilon(z)| = 0 \quad \text{and} \quad \Phi_k(x, y) = 0,$$

where

$$\Phi_k(x, y) := \frac{(\zeta(x + y_k) - 1)w_\varepsilon(x) - (\zeta(y + y_k) - 1)w_\varepsilon(y)}{|x - y|^{\frac{N}{2} + s}}.$$

Moreover, we have the following estimates:

$$\begin{aligned} |\zeta(z + y_k)w_\varepsilon(z) - w_\varepsilon(z)|^2 &\leq 2|w_\varepsilon(z)|^2, \\ |\nabla(\zeta(z + y_k)w_\varepsilon(z) - w_\varepsilon(z))|^2 &\leq C(|w_\varepsilon(z)|^2 + |\nabla w_\varepsilon(z)|^2) \end{aligned}$$

and

$$|\Phi_k(x, y)| \leq \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x - y|^{\frac{N}{2} + s}} + \frac{C|w_\varepsilon(y)|}{|x - y|^{\frac{N}{2} + s - 1}} \chi_{B(y, 1)}(x) + \frac{2|w_\varepsilon(y)|}{|x - y|^{\frac{N}{2} + s}} \chi_{B^c(y, 1)}(x).$$

Applying the Dominated Convergence Theorem, we deduce that

$$(4.2) \quad \lim_{k \rightarrow +\infty} \|\zeta(\cdot)w_\varepsilon(\cdot - y_k) - w_\varepsilon(\cdot - y_k)\|_\varepsilon^2.$$

We now claim that

$$(4.3) \quad \lim_{k \rightarrow +\infty} t_k = 1.$$

Indeed, since $t_k \varphi_k \in \mathcal{N}_\varepsilon$, it holds that

$$\|\varphi_k\|_\varepsilon^2 = t_k^{p-2} \int_\Omega |\varphi_k|^p dx.$$

On the other hand, since $w_\varepsilon \in \mathcal{N}_{\varepsilon, \infty}$, using (4.2) and the fact that

$$\int_\Omega |\varphi_k|^p dx \rightarrow \int_\Omega |w_\varepsilon|^p dx \quad \text{as } k \rightarrow \infty,$$

we conclude that $t_k \rightarrow 1$ establishing (4.3).

Therefore, by (4.2) and (4.3),

$$\lim_{k \rightarrow +\infty} \mathcal{J}_{\varepsilon, \infty}(t_k \varphi_k) = \mathcal{J}_{\varepsilon, \infty}(w_\varepsilon) = \mu_{\varepsilon, \infty}.$$

Hence, from (4.1), we conclude that

$$(4.4) \quad \mu_\varepsilon \leq \mu_{\varepsilon, \infty}.$$

Combining the initial observation $\mu_{\varepsilon, \infty} \leq \mu_\varepsilon$ with (4.4), we obtain $\mu_{\varepsilon, \infty} = \mu_\varepsilon$, completing the proof. $\square$

Theorem 4.2. *The problem (P_ε) does not admit a ground state solution.*

Proof. Suppose, by contradiction, that there exists $0 \leq v \in \mathcal{X}^{1,2}(\Omega)$ such that

$$\mathcal{J}'_\varepsilon(v) = 0 \quad \text{and} \quad \mathcal{J}_\varepsilon(v) = \inf_{u \in \mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u) = \mu_\varepsilon = \mu_{\varepsilon,\infty}$$

Extend v by zero outside Ω , so that $v \in H^1(\mathbb{R}^N)$. Then,

$$\mathcal{J}'_{\varepsilon,\infty}(v) = 0 \quad \text{with} \quad \mathcal{J}_{\varepsilon,\infty}(v) = \mu_{\varepsilon,\infty},$$

which implies that v is a ground state solution to $(P_{\varepsilon,\infty})$.

However, since $v \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, for any $x_0 \in \mathbb{R}^N \setminus \Omega$ we have that $v(x_0) = 0$, and therefore

$$\Delta v(x_0) \geq 0 \quad \text{and} \quad (-\Delta)^s v(x_0) < 0.$$

Hence,

$$0 = v^p(x_0) = -\Delta v(x_0) + \varepsilon(-\Delta)^s v(x_0) + v(x_0) < 0,$$

a contradiction. Therefore, no ground state solution exists for (P_ε) . $\square$

5. TECHNICAL RESULTS

In this section, we establish several auxiliary results that are essential for the proof of our main theorem. We begin our analysis by proving the following convergence result.

Lemma 5.1. *The following limit holds*

$$\lim_{\varepsilon \rightarrow 0^+} \mu_{\varepsilon,\infty} = \mu_\infty.$$

Proof. Let $\psi \in \mathcal{N}$ be such that

$$\mathcal{J}_\infty(\psi) = \mu_\infty \quad \text{and define} \quad t_\varepsilon := \left(\frac{\|\psi\|_\varepsilon}{\int_{\mathbb{R}^N} |\psi|^p dx} \right)^{\frac{1}{p-2}}.$$

Then $t_\varepsilon \psi \in \mathcal{N}_{\varepsilon,\infty}$ and it is straightforward to verify that $t_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$. Consequently,

$$(5.1) \quad \mu_\infty = \mathcal{J}_\infty(\psi) = \lim_{\varepsilon \rightarrow 0^+} \mathcal{J}_{\varepsilon,\infty}(\psi) \geq \limsup_{\varepsilon \rightarrow 0^+} \mu_{\varepsilon,\infty}.$$

This inequality implies that the sequence $\{\mu_{\varepsilon,\infty}\}$ is bounded. As a result, the corresponding sequence $\{w_\varepsilon\}_{\varepsilon > 0}$ is bounded in $H_{rad}^1(\mathbb{R}^N)$, due to

$$\left(\frac{1}{2} - \frac{1}{p} \right) \|w_\varepsilon\|_\varepsilon^2 = \mu_{\varepsilon,\infty} \quad \forall \varepsilon > 0,$$

Let $\{\varepsilon_k\}_k$ be a sequence such that $\varepsilon_k \rightarrow 0^+$ as $k \rightarrow \infty$. Then, for the corresponding sequence $\{\varepsilon_k\}_k$, there is $w_0 \in H_{rad}^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$w_{\varepsilon_k} \rightharpoonup w_0 \text{ in } H_{rad}^1(\mathbb{R}^N) \quad \text{and} \quad w_{\varepsilon_k} \rightarrow w_0 \text{ in } L^p(\mathbb{R}^N).$$

Since $\mathcal{J}'_{\varepsilon_k,\infty}(w_{\varepsilon_k}) = 0$, it follows that

$$\lim_{k \rightarrow \infty} \varepsilon_k \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_{\varepsilon_k}(x) - w_{\varepsilon_k}(y)|^2}{|x - y|^{N+2s}} dx dy = 0$$

and

$$(5.2) \quad \lim_{k \rightarrow \infty} \|w_{\varepsilon_k}\|_{H^1(\mathbb{R}^N)}^2 = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |w_{\varepsilon_k}|^p dx = \int_{\mathbb{R}^N} |w_0|^p dx.$$

Define

$$t_k = \left(\frac{\|w_{\varepsilon_k}\|_{H^1(\mathbb{R}^N)}^2}{\int_{\mathbb{R}^N} |w_{\varepsilon_k}|^p dx} \right)^{\frac{1}{p-2}}.$$

Then $t_k w_{\varepsilon_k} \in \mathcal{N}_\infty$ and

$$(5.3) \quad \lim_{k \rightarrow \infty} t_k = 1.$$

Using this, we estimate:

$$\begin{aligned} \mu_\infty &\leq \mathcal{J}_\infty(t_{\varepsilon_k} w_{\varepsilon_k}) \\ &= \mathcal{J}_{\varepsilon_k, \infty}(t_{\varepsilon_k} w_{\varepsilon_k}) - \frac{\varepsilon_k}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|(t_{\varepsilon_k} w_{\varepsilon_k})(x) - (t_{\varepsilon_k} w_{\varepsilon_k})(y)|^2}{|x - y|^{N+2s}} dy dx \\ &\leq t_k^2 \mathcal{J}_{\varepsilon_k, \infty}(w_{\varepsilon_k}) + (t_k^2 - t_k^p) \int_{\mathbb{R}^N} |w_{\varepsilon_k}|^p dx \\ &= t_k^2 \mu_{\varepsilon_k, \infty} + (t_k^2 - t_k^p) \int_{\mathbb{R}^N} |w_{\varepsilon_k}|^p dx. \end{aligned}$$

From this estimate, together with (5.2) and (5.3), we conclude that

$$\mu_\infty \leq \liminf_{\varepsilon \rightarrow 0^+} \mu_{\varepsilon, \infty}.$$

Combining this with (5.1), we obtain the desired limit

$$\lim_{\varepsilon \rightarrow 0^+} \mu_{\varepsilon, \infty} = \mu_\infty$$

□

We now turn our attention to the behavior of weak limits of $(PS)_c$ sequences associated with the functional $\mathcal{J}_\varepsilon$ as ε becomes small. This analysis plays a crucial role in our approach.

Proposition 5.2. *For each $\delta \in (0, \frac{\mu_\infty}{2})$, there exists $\varepsilon_1 > 0$, such that if $\{u_k\}_k$ is a nonnegative $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$ with $\varepsilon \in (0, \varepsilon_1)$ and $c \in (\mu_\infty + \delta, 2\mu_\infty - \delta)$, then $\{u_k\}_k$ admits a weak limit $u_0 \neq 0$.*

Proof. We argue by contradiction. Suppose there exist $\delta^* \in (0, \frac{\mu_\infty}{2})$, a sequence $\varepsilon_n \rightarrow 0$ such that for each n , there is a nonnegative $(PS)_{c_n}$ sequence $\{u_k^n\}_k \subset \mathcal{X}^{1,2}(\Omega)$ for $\mathcal{J}_{\varepsilon_n}$ with

$$c_n \in (\mu_\infty + \delta^*, 2\mu_\infty - \delta^*),$$

but every such sequence $\{u_k^n\}_k$ has a weak limit equal to zero. Without loss of generality, we may assume

$$(5.4) \quad c_n \rightarrow \mu_0 \in [\mu_\infty + \delta^*, 2\mu_\infty - \delta^*].$$

Since $\{u_k^n\}_k$ is a $(PS)_{c_n}$ sequence for $\mathcal{J}_{\varepsilon_n}$, we have

$$(5.5) \quad \mathcal{J}_{\varepsilon_n}(u_k^n) \rightarrow c_n \quad \text{and} \quad \mathcal{J}'_{\varepsilon_n}(u_k^n) \rightarrow 0 \quad \text{as} \quad k \rightarrow +\infty.$$

Using the definition of $\mathcal{J}_\varepsilon$, we get

$$\limsup_{k \rightarrow +\infty} \|u_k^n\|_{\varepsilon_n}^2 \leq \frac{c_n}{\left(\frac{1}{2} - \frac{1}{p}\right)} \leq \frac{2\mu_\infty - \delta^*}{\left(\frac{1}{2} - \frac{1}{p}\right)} \quad \forall n \in \mathbb{N}.$$

Hence, we can assume that the sequence $\{u_k^n\}_{k,n}$ is bounded in $\mathcal{X}^{1,2}(\Omega)$.

Claim 1. There exist $\nu > 0$, $R > 0$ and $\tilde{n} \in \mathbb{N}$ satisfying

$$\limsup_{k \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,R)} |u_k^n|^2 dx \geq \nu, \quad \forall n \geq \tilde{n}.$$

Otherwise, for each $R > 0$ and $j \in \mathbb{N}$, there exists a sequence $\{n_j\}_j \subset \mathbb{N}$, with $n_j \rightarrow \infty$, such that

$$\limsup_{k \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,R)} |u_k^{n_j}|^2 dx < \frac{1}{j}.$$

Using the argument from [41, Lemma 1.21], this implies

$$\limsup_{k \rightarrow +\infty} \int_{\Omega} |u_k^{n_j}|^q dx = o_j(1), \quad \forall q \in (2, 2^*).$$

Since $\mathcal{J}'_{\varepsilon_{n_j}}(u_k^{n_j}) = o_k(1)$, and $\{u_k^{n_j}\}_{k,j}$ is bounded in $\mathcal{X}^{1,2}(\Omega)$, we have

$$\|u_k^{n_j}\|_{\varepsilon_j}^2 = \int_{\Omega} |u_k^{n_j}|^p dx + o_k(1),$$

and hence

$$\limsup_{k \rightarrow +\infty} \|u_k^{n_j}\|_{\varepsilon_j}^2 = o_j(1).$$

It follows that

$$\mu_{\infty} + \delta^* < c_{n_j} = \limsup_{k \rightarrow +\infty} \mathcal{J}_{\varepsilon_{n_j}}(u_k^{n_j}) \leq \frac{1}{2} \limsup_{k \rightarrow +\infty} \|u_k^{n_j}\|_{\varepsilon_j}^2 = o_j(1),$$

a contradiction.

By Claim 1, for each n , we can find a subsequence of $\{u_k^n\}_k$ and a sequence $\{z_k^n\}_k \subset \mathbb{R}^N$ such that

$$(5.6) \quad \int_{B(z_k^n, R)} |u_k^n|^2 dx \geq \frac{\nu}{2}, \quad \forall k, n \in \mathbb{N}.$$

Claim 2. For each n , we have

$$\lim_{k \rightarrow +\infty} |z_k^n| = +\infty.$$

Indeed, if $\{z_k^n\}_k$ is bounded, there is $R_n > 0$ such that

$$\int_{B(0, R_n)} |u_k^n|^2 dx \geq \int_{B(z_k^n, R)} |u_k^n|^2 dx \geq \frac{\nu}{2} > 0.$$

However, since $u_k^n \rightharpoonup 0$ in $\mathcal{X}^{1,2}(\Omega)$, we must have

$$\lim_{k \rightarrow +\infty} \int_{B(0, R_n)} |u_k^n|^2 dx = 0,$$

which is a contradiction.

Combining **Claim 2** with (5.5) and (5.6), for each $n \in \mathbb{N}$, we can choose k_n such that

$$n < |z_{k_n}^n|, \quad \frac{\nu}{2} \leq \int_{B(z_{k_n}^n, R)} |u_{k_n}^n|^2 dx, \quad |\mathcal{J}_{\varepsilon_n}(u_{k_n}^n) - c_n| < \frac{1}{n} \quad \text{and} \quad \|\mathcal{J}'_{\varepsilon_n}(u_{k_n}^n)\| < \frac{1}{n}$$

Define $u_n := u_{k_n}^n$, $z_n := z_{k_n}^n$. Then,

$$n < |z_n|, \quad \frac{\nu}{2} \leq \int_{B(z_n, R)} |u_n|^2 dx, \quad |\mathcal{J}_{\varepsilon_n}(u_n) - c_n| < \frac{1}{n} \quad \text{and} \quad \|\mathcal{J}'_{\varepsilon_n}(u_n)\| < \frac{1}{n}.$$

Note that the sequence $\{u_n\}$ is bounded in $\mathcal{X}^{1,2}(\Omega)$ and satisfies

$$|\mathcal{J}_{\varepsilon_n}(u_n) - \mathcal{I}(u_n)| = \frac{\varepsilon_n}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dy dx = o_n(1)$$

and

$$\begin{aligned} \sup_{\|\varphi\|_{H_0^1(\Omega)} \leq 1} |\mathcal{J}'_{\varepsilon_n}(u_n)\varphi - \mathcal{I}'(u_n)\varphi| &\leq \sup_{\|\varphi\|_{H_0^1(\Omega)} \leq 1} \varepsilon_n \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[u_n(x) - u_n(y)][\varphi(x) - \varphi(y)]}{|x - y|^{N+2s}} dy dx \\ &= o_n(1). \end{aligned}$$

Hence, using (5.4),

$$\mathcal{I}(u_n) \rightarrow \mu_0, \quad \text{and} \quad \mathcal{I}'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Up to a subsequence, $u_n \rightharpoonup u_0$ in $\mathcal{X}^{1,2}(\Omega)$ and by standard argument, $\mathcal{I}'(u_0) = 0$.

If $u_n \rightarrow u_0$ in $\mathcal{X}^{1,2}(\Omega)$, then since $|z_n| \rightarrow \infty$, we would get

$$\int_{B(z_n, R)} |u_0|^2 dx < \frac{\nu}{4}, \quad \text{and} \quad \left| \int_{B(z_n, R)} |u_n|^2 dx - \int_{B(z_n, R)} |u_0|^2 dx \right| < \frac{\nu}{4}$$

for n large enough. Then, for n large enough

$$\frac{\nu}{2} \leq \int_{B(z_n, R)} |u_n|^2 dx < \frac{\nu}{2},$$

which is a contradiction.

Thus $u_n \not\rightarrow u_0$, and by [6, Lemma 3.1] there exist $k \in \mathbb{N}$, $k \geq 1$ and $v_1, v_2, \dots, v_m$ solutions of problem (P_∞) such that

$$(5.7) \quad \mathcal{I}(u_n) = \mathcal{I}(u_0) + \sum_{j=1}^m \mathcal{J}_\infty(v_j) + o_n(1)$$

If $u_0 = 0$, then (5.7) implies

$$\mathcal{I}(u_n) \geq (m+1)\mu_\infty + o_n(1)$$

which ensures that $\mu_0 \geq (m+1)\mu_\infty \geq 2\mu_\infty$, which is a contradiction. Therefore, $u_0 = 0$, and (5.7) gives $\mu_0 \geq m\mu_\infty$. So, $m = 1$, $v_1 > 0$ and

$$\mathcal{I}(u_n) = \mathcal{I}(v_1) + o_n(1) = \mu_\infty + o_n(1),$$

yielding $\mu_0 = \mu_\infty$, which contradicts (5.4). This completes the proof. $\square$

As a consequence of Proposition 5.2, we can show the following result.

Proposition 5.3. *For every $\delta \in (0, \frac{\mu_\infty}{2})$, there exists $\varepsilon_1 > 0$ such that, if $\{u_k\}_{k \in \mathbb{N}}$ is a nonnegative $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$ restricted to $\mathcal{N}_\varepsilon$ with $c \in (\mu_\infty + \delta, 2\mu_\infty - \delta)$ and $\varepsilon \in (0, \varepsilon_1)$, then there exists $u_0 \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\}$ such that $u_k \rightharpoonup u_0$ in $\mathcal{X}^{1,2}(\Omega)$.*

Proof. Let $\{u_k\}_k \subset \mathcal{N}_\varepsilon$ be such that

$$(5.8) \quad \mathcal{J}_\varepsilon(u_k) \rightarrow c \in (\mu_\infty + \delta, 2\mu_\infty - \delta) \quad \text{and} \quad \|\mathcal{J}'_\varepsilon(u_k)\|_* \rightarrow 0, \quad \text{as } k \rightarrow +\infty,$$

where

$$\|\mathcal{J}'_\varepsilon(u)\|_* = \sup \{ |\mathcal{J}'_\varepsilon(u)\varphi| : \varphi \in T_u \mathcal{N}_\varepsilon \quad \text{and} \quad \|\varphi\|_\varepsilon \leq 1 \}.$$

Then, since $\{u_k\}_k \subset \mathcal{N}_\varepsilon$, we obtain

$$(5.9) \quad c = \lim_{k \rightarrow \infty} \mathcal{J}_\varepsilon(u_k) = \lim_{k \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|_\varepsilon^2.$$

On the other hand, there is a sequence $\{\lambda_k\} \subset \mathbb{R}$ such that

$$(5.10) \quad \mathcal{J}'_\varepsilon(u_k) = \lambda_k \mathcal{L}'_\varepsilon(u_k) + o_k(1),$$

where

$$\mathcal{L}_\varepsilon(u) = \mathcal{J}'_\varepsilon(u)u = \|u\|_\varepsilon^2 - \int_\Omega |u|^p dx.$$

Since $\{u_k\}_k \subset \mathcal{N}_\varepsilon$ and by (5.9), we get

$$\lim_{k \rightarrow \infty} \mathcal{L}'_\varepsilon(u_k)u_k = \lim_{k \rightarrow \infty} \mathcal{L}'_\varepsilon(u_k)u_k - p\mathcal{L}_\varepsilon(u_k) = \lim_{k \rightarrow \infty} (2-p)\|u_k\|_\varepsilon^2 = -2pc \neq 0.$$

Therefore, from (5.8) and (5.10), it follows that $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$.

Furthermore, (5.10) implies that $\|\mathcal{J}'_\varepsilon\| \rightarrow 0$ as $k \rightarrow \infty$, so $(u_k)_{k \in \mathbb{N}}$ is a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$ in $\mathcal{X}^{1,2}(\Omega)$.

The conclusion now follows directly from Proposition 5.2. $\square$

For $R > 0$, define the function

$$\Upsilon(t) := \begin{cases} 1, & t \in [0, R], \\ \frac{R}{t}, & t > R. \end{cases}$$

Define the barycentric function

$$\tau: H^1(\mathbb{R}^N) \rightarrow \mathbb{R}^N, \quad \tau(u) := \int_{\mathbb{R}^N} |u(x)|^2 \Upsilon(|x|) dx,$$

and the sets

$$\begin{aligned} \mathcal{P} &:= \{u \in \mathcal{X}^{1,2}(\Omega) : u \geq 0\}, \\ \mathcal{T}_0^\varepsilon &:= \{u \in \mathcal{N}_\varepsilon \cap \mathcal{P} : \tau(u) = 0\}, \\ \mathcal{T}_0 &:= \{u \in \mathcal{N} \cap \mathcal{P} : \tau(u) = 0\}, \end{aligned}$$

and the real numbers

$$c_0^\varepsilon := \inf_{u \in \mathcal{T}_0^\varepsilon} \mathcal{J}_\varepsilon(u) \quad \text{and} \quad c_0 := \inf_{u \in \mathcal{T}_0} \mathcal{I}(u).$$

Clearly,

$$c_0^\varepsilon \geq \mu_\varepsilon \geq \beta \quad \text{and} \quad c_0 \geq \mu \geq \beta,$$

where β is the constant given by the first geometric condition of the Mountain Pass Theorem.

Lemma 5.4. *The following limit holds:*

$$\lim_{\varepsilon \rightarrow 0} c_0^\varepsilon = c_0$$

Proof. For each $\varepsilon > 0$, fix $v_\varepsilon \in \mathcal{T}_0^\varepsilon$ such that

$$\mathcal{J}_\varepsilon(v_\varepsilon) \leq c_0^\varepsilon + \varepsilon$$

and let $\sigma_\varepsilon > 0$ be such that $\sigma_\varepsilon v_\varepsilon \in \mathcal{N}$. Then $\sigma_\varepsilon v_\varepsilon \in \mathcal{T}_0$, and we obtain

$$(5.11) \quad c_0 \leq \mathcal{I}(\sigma_\varepsilon v_\varepsilon) \leq \mathcal{J}_\varepsilon(\sigma_\varepsilon v_\varepsilon) \leq \max_{\sigma \geq 0} \mathcal{J}_\varepsilon(\sigma v_\varepsilon) = \mathcal{J}_\varepsilon(v_\varepsilon) \leq c_0^\varepsilon + \varepsilon.$$

On the other hand, fix $u_\varepsilon \in \mathcal{T}_0$ such that

$$\mathcal{I}(u_\varepsilon) \leq c_0 + \varepsilon \leq c_0^\varepsilon + 2\varepsilon,$$

and let $\kappa_\varepsilon > 0$ be such that $\kappa_\varepsilon u_\varepsilon \in \mathcal{N}_\varepsilon$. Then $\kappa_\varepsilon u_\varepsilon \in \mathcal{T}_0^\varepsilon$ and we have

$$\begin{aligned}
 (5.12) \quad c_0^\varepsilon &\leq \mathcal{J}_\varepsilon(\kappa_\varepsilon u_\varepsilon) = \mathcal{I}(\kappa_\varepsilon u_\varepsilon) + \frac{\varepsilon}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|(\kappa_\varepsilon u_\varepsilon)(x) - (\kappa_\varepsilon u_\varepsilon)(y)|^2}{|x - y|^{N+2s}} dy dx \\
 &\leq \mathcal{I}(u_\varepsilon) + \frac{\varepsilon \kappa_\varepsilon^2}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{N+2s}} dy dx \\
 &\leq c_0 + \varepsilon + \frac{\varepsilon \kappa_\varepsilon^2}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{N+2s}} dy dx.
 \end{aligned}$$

Since $u_\varepsilon \in \mathcal{N}$, we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) \|u_\varepsilon\|_{H_0^1(\Omega)}^2 = \mathcal{I}(u_\varepsilon) - \frac{1}{p} \mathcal{I}'(u_\varepsilon) u_\varepsilon \leq c_0 + \varepsilon,$$

which shows that $\|u_\varepsilon\|_{H_0^1(\Omega)}^2$ is bounded. Moreover, by the mountain pass geometry of $\mathcal{I}$, it follows that

$$\|u_\varepsilon\|_{H_0^1(\Omega)}^2 \rightharpoonup 0 \quad \text{and} \quad \|u_\varepsilon\|_{L^p(\Omega)}^p \rightharpoonup 0.$$

These facts, together with the condition $\kappa_\varepsilon u_\varepsilon \in \mathcal{N}_\varepsilon$, imply that the sequence (κ_ε) is bounded. Hence, there exists a constant $\mathcal{K} > 0$ such that

$$\frac{\kappa_\varepsilon^2}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{N+2s}} dy dx \leq \mathcal{K}.$$

Therefore, from (5.12), we obtain

$$(5.13) \quad c_0^\varepsilon \leq c_0 + o_\varepsilon(1).$$

Combining inequalities (5.11) and (5.13), we conclude that

$$\lim_{\varepsilon \rightarrow 0} c_0^\varepsilon = c_0.$$

□

Lemma 5.5. *The inequality $\mu_\infty < c_0$ holds.*

Proof. Define the set

$$\mathcal{T} := \{u \in \mathcal{X}^{1,2}(\Omega) \cap \mathcal{P} : \|u\|_{L^p(\Omega)} = 1, \tau(u) = 0\}$$

and set

$$\tilde{c}_0 := \inf_{u \in \mathcal{T}} \|u\|_{H_0^1(\Omega)}^2.$$

Note that, if $u \in \mathcal{T}_0$, then the normalized function $\frac{u}{\|u\|_{L^p(\Omega)}} \in \mathcal{T}$ and

$$\tilde{c}_0 \leq \left\| \frac{u}{\|u\|_{L^p(\Omega)}} \right\|_{H_0^1(\Omega)}^2.$$

This implies

$$\|u\|_{L^p(\Omega)}^p \tilde{c}_0^{\frac{p}{2}} \leq \|u\|_{H_0^1(\Omega)}^p.$$

Since $u \in \mathcal{T}_0$, we also have $\|u\|_{H_0^1(\Omega)}^2 = \|u\|_{L^p(\Omega)}^p$, and so

$$\|u\|_{H_0^1(\Omega)}^2 \tilde{c}_0^{\frac{p}{2}} \leq \|u\|_{H_0^1(\Omega)}^p,$$

which yields

$$\tilde{c}_0^{\frac{p}{p-2}} \leq \|u\|_{H_0^1(\Omega)}^2.$$

Hence, for all $u \in \mathcal{T}_0$,

$$\mathcal{I}(u) = \mathcal{I}(u) - \frac{1}{p} \mathcal{I}'(u)u = \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_{H_0^1(\Omega)}^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right) \tilde{c}_0^{\frac{p}{p-2}}.$$

Therefore,

$$(5.14) \quad \left(\frac{1}{2} - \frac{1}{p}\right) \tilde{c}_0^{\frac{p}{p-2}} \leq c_0$$

On the other hand, consider

$$\mathcal{M} := \inf \left\{ \|u\|_{H^1(\mathbb{R}^N)}^2 : u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^p dx = 1 \right\}$$

then

$$(5.15) \quad \mu_\infty = \mathcal{J}_\infty(\psi) = \left(\frac{1}{2} - \frac{1}{p}\right) \mathcal{M}^{\frac{p}{p-2}}.$$

By [6, Lemma 4.3], we have

$$\mathcal{M} < \tilde{c}_0.$$

Combining this with (5.14) and (5.15), we conclude

$$\mu_\infty < \left(\frac{1}{2} - \frac{1}{p}\right) \tilde{c}_0^{\frac{p}{p-2}} \leq c_0,$$

as claimed. □

Let $\eta: \mathbb{R}^+ \cup \{0\} \rightarrow [0, 1]$ be a non-decreasing function of class C^∞ such that

$$\eta(t) = 0 \quad \text{for } t \leq 1, \quad \eta(t) = 1 \quad \text{for } t \geq 2.$$

Let $\rho > 0$ be the smallest positive number such that

$$\mathbb{R}^N \setminus \Omega \subset B(0, \rho).$$

Define, for each $x \in \mathbb{R}^N$,

$$\zeta(x) := \eta\left(\frac{|x|}{\rho}\right).$$

Given $y \in \mathbb{R}^N$, consider the function $\Psi_{y,\rho} \in H^1(\mathbb{R}^N)$ defined by

$$\Psi_{y,\rho}(x) := \zeta(x) \psi(x - y),$$

where $\psi \in \mathcal{N}_\infty$ is a ground state solution of problem (P_∞) (see Section 3).

For each $y \in \mathbb{R}^N$, let $\sigma_{y,\rho}^\varepsilon > 0$ be such that

$$\varphi_{\varepsilon,\rho}(y) := \sigma_{y,\rho}^\varepsilon \Psi_{y,\rho} \in \mathcal{N}_{\varepsilon,\infty}.$$

With this notation, we have the following:

Lemma 5.6. *The following assertions hold:*

(1)

$$\lim_{\rho, \varepsilon \rightarrow 0} \mathcal{J}_{\varepsilon,\infty}(\varphi_{\varepsilon,\rho}(y)) = \mu_\infty \quad \text{uniformly in } y \in \mathbb{R}^N.$$

(2) For each fixed $\rho > 0$,

$$\lim_{\substack{|y| \rightarrow \infty, \\ \varepsilon \rightarrow 0}} \mathcal{J}_{\varepsilon,\infty}(\varphi_{\varepsilon,\rho}(y)) = \mu_\infty.$$

Proof. (1) Arguing as in [1, Lemma 4.6] and [3, Lemma 4.1], we obtain

$$(5.16) \quad \|\Psi_{y,\rho}\|_{H^1(\mathbb{R}^N)}^2 \rightarrow \|\psi\|_{H^1(\mathbb{R}^N)}^2, \quad \|\Psi_{y,\rho}\|_{L^p(\mathbb{R}^N)}^p \rightarrow \|\psi\|_{L^p(\mathbb{R}^N)}^p$$

and

$$(5.17) \quad \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\Psi_{y,\rho}(x) - \Psi_{y,\rho}(z)|^2}{|x - z|^{N+2s}} dz dx \rightarrow \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi(x) - \psi(z)|^2}{|x - z|^{N+2s}} dz dx$$

as $\rho \rightarrow 0$, uniformly in y .

Since $\sigma_{y,\rho}^\varepsilon \Psi_{y,\rho} \in \mathcal{N}_{\varepsilon,\infty}$, we have

$$\sigma_{y,\rho}^\varepsilon = \left(\frac{\|\Psi_{y,\rho}\|_\varepsilon^2}{\|\Psi_{y,\rho}\|_{L^p(\mathbb{R}^N)}^p} \right)^{\frac{1}{p-2}}.$$

Then, from (5.16) and (5.17), we conclude

$$\lim_{\rho, \varepsilon \rightarrow 0} \sigma_{y,\rho}^\varepsilon = \left(\frac{\|\psi\|_{H^1(\mathbb{R}^N)}^2}{\|\psi\|_{L^p(\mathbb{R}^N)}^p} \right)^{\frac{1}{p-2}} = 1$$

uniformly in y .

Consequently, combining this with (5.16) and (5.17), we obtain

$$\lim_{\rho, \varepsilon \rightarrow 0} \|\sigma_{y,\rho}^\varepsilon \Psi_{y,\rho}\|_\varepsilon^2 = \|\psi\|_{H^1(\mathbb{R}^N)}^2.$$

uniformly in y . Hence

$$\lim_{\rho, \varepsilon \rightarrow 0} \mathcal{J}_{\varepsilon,\infty}(\varphi_{\varepsilon,\rho}(y)) = \mathcal{J}_\infty(\psi) = \mu_\infty,$$

uniformly in y .

(2) Fix $\rho > 0$ and consider a sequence $\{y_k\}_k \subset \mathbb{R}^N$ with $|y_k| \rightarrow +\infty$. Using the same arguments as in the proof of Lemma 4.1, we get

$$\lim_{\substack{|y_k| \rightarrow +\infty, \\ \varepsilon \rightarrow 0}} \|\varphi_{\varepsilon,\rho}(y_k)\|_\varepsilon^2 = \|\psi\|_{H^1(\mathbb{R}^N)}^2 \quad \text{and} \quad \lim_{\substack{|y_k| \rightarrow +\infty, \\ \varepsilon \rightarrow 0}} \|\varphi_{\varepsilon,\rho}(y_k)\|_{L^p(\mathbb{R}^N)}^p = \|\psi\|_{L^p(\mathbb{R}^N)}^p.$$

Therefore

$$\lim_{\substack{|y_k| \rightarrow +\infty, \\ \varepsilon \rightarrow 0}} \mathcal{J}_{\varepsilon,\infty}(\varphi_{\varepsilon,\rho}(y_k)) = \mathcal{J}_\infty(\psi) = \mu_\infty.$$

□

As a consequence of Lemma 5.6-(1), we have the following result.

Lemma 5.7. *Fix $\delta \in (0, \frac{\mu_\infty}{2})$. Then, there exist $\tilde{\rho} = \tilde{\rho}(\delta)$ and $\varepsilon_2 = \varepsilon_2(\delta)$ such that*

$$\sup_{y \in \mathbb{R}^N} \mathcal{J}_{\varepsilon,\infty}(\varphi_{\varepsilon,\rho}(y)) < 2\mu_\infty - \delta, \quad \text{for all } \rho < \tilde{\rho} \text{ and } \varepsilon < \varepsilon_2.$$

In what follows, we fix $\delta \in (0, \frac{\mu_\infty}{2})$ such that $\mu_\infty + \delta < c_0$. Also, we fix $\rho < \tilde{\rho}$ and Ω such that

$$\mathbb{R}^N \setminus \Omega \subset B(0, \rho).$$

Lemma 5.8. *There exist $\varepsilon_3 > 0$, $\tilde{R} > \rho$ such that*

- (1) $\mu_{\varepsilon,\infty} < \mathcal{J}_\varepsilon(\varphi_{\varepsilon,\rho}(y)) < \frac{c_0^\varepsilon + \mu_{\varepsilon,\infty}}{2}$ for all $|y| \geq \tilde{R}$ and $\varepsilon \in (0, \varepsilon_3)$,
- (2) $\langle \tau(\varphi_{\varepsilon,\rho}(y)), y \rangle > 0$ for $|y| = \tilde{R}$.

Proof. (1) Since $\mathbb{R}^N \setminus \Omega \subset B(0, \rho)$, by the definition of $\varphi_{\varepsilon, \rho}$ we have $\varphi_{\varepsilon, \rho}(y) \in \mathcal{N}_\varepsilon$. Then, from Lemma 4.1 and Theorem 4.2, we obtain

$$(5.18) \quad \mu_{\varepsilon, \infty} = \mu_\varepsilon < \mathcal{J}_\varepsilon(\varphi_{\varepsilon, \rho}(y)), \quad \forall \varepsilon > 0.$$

From Lemmas 5.1, 5.4 and 5.5, we deduce that for $\varrho > 0$ sufficiently small, there exists $\varepsilon_3 > 0$ such that

$$\mu_\infty < \frac{c_0 + \mu_\infty}{2} - \varrho < \frac{c_0^\varepsilon + \mu_{\varepsilon, \infty}}{2} < \frac{c_0 + \mu_\infty}{2} + \varrho, \quad \forall \varepsilon \in (0, \varepsilon_3).$$

Moreover, from Lemma 5.6-(2), there is $\tilde{R} > \rho$ such that

$$(5.19) \quad \mathcal{J}_\varepsilon(\varphi_{\varepsilon, \rho}(y)) < \frac{c_0 + \mu_\infty}{2} - \varrho, \quad \forall \varepsilon \in (0, \varepsilon_3) \quad \text{and} \quad |y| \geq \tilde{R}.$$

Combining (5.18)–(5.19), we conclude that item (1) holds.

(2) The proof follows similarly to the argument in [6, Lemma 4.3], and is therefore omitted. $\square$

6. PROOF OF THEOREM 1.1

Our goal in this section is to prove our main result. However, before doing so, we need to introduce the following sets:

$$\begin{aligned} \mathcal{G} &:= \left\{ \varphi_{\varepsilon, \rho}(y) : |y| \leq \tilde{R} \right\}, \\ \mathcal{H} &:= \left\{ f \in C(\mathcal{P} \cap \mathcal{N}_\varepsilon, \mathcal{P} \cap \mathcal{N}_\varepsilon) : f(u) = u \text{ if } \mathcal{J}_\varepsilon(u) < \frac{c_0^\varepsilon + \mu_{\varepsilon, \infty}}{2} \right\}, \\ \Gamma &:= \{f(\mathcal{G}) : f \in \mathcal{H}\}. \end{aligned}$$

The next result is stated in [6, Lemma 4.4], and its proof follows similar arguments. For this reason, we omit it.

Lemma 6.1. *If $A \in \Gamma$, then $A \cap \mathcal{T}_0^\varepsilon \neq \emptyset$ for all $\varepsilon \in (0, \varepsilon_3)$, where ε_3 is given in Lemma 5.8.*

We are now ready to establish the existence of a higher energy positive solution.

Proof of Theorem 1.1. Define the level

$$\mathcal{C}_\varepsilon := \inf_{A \in \Gamma} \sup_{u \in A} \mathcal{J}_\varepsilon(u).$$

By Lemma 6.1, for each $A \in \Gamma$ there exists $\hat{u} \in A \cap \mathcal{T}_0^\varepsilon$. Hence

$$c_0^\varepsilon = \inf_{u \in \mathcal{T}_0^\varepsilon} \mathcal{J}_\varepsilon(u) \leq \mathcal{J}_\varepsilon(\hat{u}) \leq \sup_{u \in A} \mathcal{J}_\varepsilon(u), \quad \forall A \in \Gamma,$$

which implies

$$(6.1) \quad c_0^\varepsilon \leq \mathcal{C}_\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_3).$$

Now, let $\delta \in (0, \frac{\mu_\infty}{2})$ be such that $\mu_\infty + \delta < c_0$. By Lemma 5.4, there is $\varepsilon_4 > 0$ such that

$$(6.2) \quad \mu_\infty + \delta < c_0^\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_4).$$

By the definition of $\mathcal{C}_\varepsilon$, we also have

$$\mathcal{C}_\varepsilon \leq \sup_{u \in A} \mathcal{J}_\varepsilon(u), \quad \forall A \in \Gamma,$$

and in particular,

$$\mathcal{C}_\varepsilon \leq \sup_{\varphi_{\varepsilon, \rho}(y) \in \mathcal{G}} \mathcal{J}_\varepsilon(f(\varphi_{\varepsilon, \rho}(y))), \quad \forall f \in \mathcal{H}.$$

Taking $f = I_{\mathbb{R}^N}$ and using Lemma 5.7 we obtain

$$(6.3) \quad \mathcal{C}_\varepsilon \leq \sup_{\varphi_{\varepsilon,\rho}(y) \in \mathcal{G}} \mathcal{J}_\varepsilon(\varphi_{\varepsilon,\rho}(y)) = \sup_{|y| \leq \tilde{R}} \mathcal{J}_\varepsilon(\varphi_{\varepsilon,\rho}(y)) < 2\mu_\infty - \delta, \quad \forall \varepsilon \in (0, \varepsilon_2).$$

From (6.1), (6.2) and (6.3), it follows that

$$\mu_\infty + \delta < \mathcal{C}_\varepsilon < 2\mu_\infty - \delta, \quad \forall \varepsilon < \varepsilon_5 := \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\}.$$

Therefore, there exists $\kappa > 0$ such that

$$0 < \kappa < \mathcal{C}_\varepsilon - \frac{c_0^\varepsilon + \mu_{\varepsilon,\infty}}{2}.$$

We claim that for each $\kappa \in \left(0, \mathcal{C}_\varepsilon - \frac{c_0^\varepsilon + \mu_{\varepsilon,\infty}}{2}\right)$, there exists

$$u_\kappa \in (\mathcal{P} \cap \mathcal{N}_\varepsilon) \cap \mathcal{J}_\varepsilon^{-1}(\mathcal{C}_\varepsilon - \kappa, \mathcal{C}_\varepsilon + \kappa)$$

such that

$$\|\mathcal{J}'_\varepsilon(u_\kappa)\|_* < \kappa,$$

where

$$\|\mathcal{J}'_\varepsilon(u)\|_* = \sup \left\{ |\mathcal{J}'_\varepsilon(u)\varphi| : \varphi \in T_u \mathcal{N}_\varepsilon \text{ and } \|\varphi\|_\varepsilon \leq 1 \right\}, \quad \forall u \in \mathcal{N}_\varepsilon.$$

Assume by contradiction that this is false. Then there exists $\kappa_0 > 0$ with $\kappa_0 < \mathcal{C}_\varepsilon - \frac{c_0^\varepsilon + \mu_{\varepsilon,\infty}}{2}$ such that

$$\|\mathcal{J}'_\varepsilon(u)\|_* \geq \frac{\kappa_0}{2}, \quad \forall u \in \mathcal{J}_\varepsilon^{-1}(\mathcal{C}_\varepsilon - \kappa_0, \mathcal{C}_\varepsilon + \kappa_0) \cap (\mathcal{P} \cap \mathcal{N}_\varepsilon).$$

By the Deformation Lemma (see [41, Lemma 5.15]), there exists $\mathcal{H} : [0, 1] \times \mathcal{P} \cap \mathcal{N}_\varepsilon \rightarrow \mathcal{P} \cap \mathcal{N}_\varepsilon$ such that

$$\mathcal{H}(t, u) = u, \quad \text{if } u \notin \mathcal{J}_\varepsilon^{-1}(\mathcal{C}_\varepsilon - \kappa_0, \mathcal{C}_\varepsilon + \kappa_0)$$

and

$$(6.4) \quad \mathcal{H} \left(1, \mathcal{J}_\varepsilon^{\mathcal{C}_\varepsilon + \frac{\kappa_0}{2}} \right) \subset \mathcal{J}_\varepsilon^{\mathcal{C}_\varepsilon - \frac{\kappa_0}{2}},$$

where

$$\mathcal{J}_\varepsilon^\gamma := \{u \in \mathcal{P} \cap \mathcal{N}_\varepsilon : \mathcal{J}_\varepsilon(u) \leq \gamma\}.$$

According to the definition of $\mathcal{C}_\varepsilon$, we have

$$\sup_{u \in \tilde{A}} \mathcal{J}_\varepsilon(u) \leq \mathcal{C}_\varepsilon + \frac{\kappa_0}{2}, \quad \text{for some } \tilde{A} \in \Gamma.$$

Hence $\tilde{A} \subset \mathcal{J}_\varepsilon^{\mathcal{C}_\varepsilon + \frac{\kappa_0}{2}}$, and from (6.4)

$$\mathcal{H}(1, \tilde{A}) \subset \mathcal{J}_\varepsilon^{\mathcal{C}_\varepsilon - \frac{\kappa_0}{2}},$$

implying

$$(6.5) \quad \sup_{u \in \mathcal{H}(1, \tilde{A})} \mathcal{J}_\varepsilon(u) \leq \mathcal{C}_\varepsilon - \frac{\kappa_0}{2}.$$

Since $\tilde{A} \in \Gamma$, there is $f \in \mathcal{H}$ such that $\tilde{A} = f(\mathcal{G})$. Define

$$\hat{h} := \mathcal{H}(1, \cdot) \circ f \in C(\mathcal{P} \cap \mathcal{N}_\varepsilon, \mathcal{P} \cap \mathcal{N}_\varepsilon).$$

Note that for any u such that

$$\mathcal{J}_\varepsilon(u) < \frac{c_0^\varepsilon + \mu_{\varepsilon,\infty}}{2} < \mathcal{C}_\varepsilon - \kappa_0,$$

we have

$$f(u) = u \quad \text{and} \quad \mathcal{H}(1, u) = u,$$

so $\hat{h} \in \mathcal{H}$, and

$$\mathcal{H}(1, \tilde{A}) = \mathcal{H}(1, f(\mathcal{G})) = \hat{h}(\mathcal{G}) \in \Gamma.$$

Therefore

$$\mathcal{C}_\varepsilon \leq \sup_{u \in \mathcal{H}(1, \tilde{A})} \mathcal{J}_\varepsilon(u),$$

which contradicts (6.5). Thus, our claim holds.

We conclude that there exists $u_k \in \mathcal{J}_\varepsilon^{-1}(\mathcal{C}_\varepsilon - \kappa_k, \mathcal{C}_\varepsilon + \kappa_k) \cap (\mathcal{P} \cap \mathcal{N}_\varepsilon)$, with $\kappa_k \rightarrow 0$ such that

$$\|\mathcal{J}'_\varepsilon(u_k)\|_* < \kappa_k, \quad \forall k \in \mathbb{N},$$

that is, $\{u_k\}_k \subset \mathcal{P} \cap \mathcal{N}_\varepsilon$ is a $(PS)_{\mathcal{C}_\varepsilon}$ sequence for $\mathcal{J}_\varepsilon$ with

$$\mathcal{C}_\varepsilon \in (\mu_\infty + \delta, 2\mu_\infty - \delta).$$

Proposition 5.3 ensures that, there is $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, $\{u_k\}$ has a weak limit $u_\varepsilon \neq 0$ and

$$\mathcal{J}'_\varepsilon(u_\varepsilon) = 0.$$

Moreover, since $(u_k) \subset \mathcal{P}$, we also have u_ε is a nonnegative function. Therefore, there is $\varepsilon_* > 0$ such that for any $\varepsilon \in (0, \varepsilon_*)$, problem (P_ε) has at least one positive solution $u_\varepsilon \in \mathcal{X}^{1,2}(\Omega)$ with

$$\mathcal{J}_\varepsilon(u_\varepsilon) \in [\mu_\infty + \delta, 2\mu_\infty - \delta],$$

concluding the proof of Theorem. $\square$

7. DECLARATIONS

Ethical Approval: Not applicable

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