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Analysis and reliability of separable systems

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ABSTRACT

The operation of a system, such as a vehicle, communication network or automatic process, heavily depends on the correct operation of its components. A Stochastic Binary System (SBS) mathematically models the behavior of on-off systems, where the components are subject to probabilistic failures. Our goal is to understand the reliability of the global system.

The reliability evaluation of an SBS belongs to the class of NP-Hard problems, and the combinatorics of SBS imposes several challenges. In a previous work by the same authors, a special sub-class of SBSs called *separable systems* was introduced. These systems accept an efficient representation by a linear inequality on the binary states of the components. However, the reliability evaluation of separable systems is still hard.

A theoretical contribution in the understanding of separable systems is given. We fully characterize separable systems under the all-terminal reliability model, finding that they admit efficient reliability evaluation in this relevant context.

1. Introduction

Research on system reliability includes different models, metrics and algorithms for analyzing how the possible failures of components affect the behavior of a complex system. The practical applications of system reliability analysis are steadily growing in frequency and diversity. We mention just a few examples here. For instance, Johansson et al. [1] considers a joint reliability/vulnerability analysis of critical infrastructures, with a practical impact into electrical networks. The article by Macchi et al. [2] develops a reliability model in order to identify critical elements in a railway system. The results have practical use in the Italian public company Rete Ferroviaria Italiana. The work by Li et al. [3] discusses two measures of infrastructure networks increasingly dependent on information systems, namely connectivity reliability and the topological controllability in terms of topology, robustness, and node importance, taking eight city-level power transmission networks and thousands of artificial networks to discuss the use of these measures to improve reliability-based design. The work by Muriel-Villegas et al. [4] analyzes the impact of natural hazards on transportation networks in Colombia, focusing on the connectivity reliability and vulnerability of inter-urban transportation affecting remote populations in the case of disasters such as floods.

Classical network analysis was developed taking into account a system modeled as a graph, where either nodes or links (or both) are subject to failure, and where the system as a whole works correctly when the subgraph resulting from deleting the failed components verifies some connectivity constraint (in general, that a given subset of nodes is connected; this includes all terminal connectivity and source-terminal connectivity as special cases). The theory of stochastic binary systems can be seen as a generalization of network reliability models; as the system structure function in an SBS can be any arbitrary boolean function of the components, instead of a variant of a connectivity function over a graph.

The mathematical understanding of SBS involves challenges in terms of complexity, combinatorics, and reliability analysis. In the most general setting, even the determination of operational or non-operational configurations are algorithmically hard problems. Under a realistic assumption of monotonicity or *well-behavior*, finding minimally operational configurations accept efficient algorithms; see [5] for details. The interplay between static systems and dynamical stochastic binary systems is also being explored, in terms of stochastic processes [6]. The lifetimes of independent random variables govern a stochastic process where the components fail, until a non-operational system is obtained. An elegant interplay between these dynamical systems provides a new approach to reliability estimation.

The reliability evaluation of SBS belongs to the class of $\mathcal{NP}$-Hard problems. Furthermore, reliability evaluation of special (well-behaved)

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SBS also belongs to this class. This fact promotes the need of distinguished sub-classes, and the development of novel approximative techniques. In [7] the concept of separability in stochastic binary systems was introduced. As discussed below, separable systems are those whose structure function can be characterized by a hyperplane separating operational from failure states. Separable systems have some particular properties which can be of interest in the study of system reliability.

This paper aims to advance in the analysis of separable system. Specifically, the contributions of the present work can be summarized as follows:

1. Useful characterization of separable systems are established.
2. A separable system under the all-terminal reliability model is called a separable graph. We fully characterize separable graphs. As a corollary, we conclude that the reliability evaluation of separable graphs can be obtained in linear time.
3. A discussion of the level of separability for non-separable systems is presented.

The remainder of this paper is organized in this way. Section 2 presents the main definitions of stochastic binary systems and separable systems. A theoretical analysis of separable systems is covered in Section 3. A particular analysis of the all-terminal reliability model is presented in Section 4. Generalizations of the concept of separability by hyperplanes are studied in Section 5. Finally, Section 6 presents concluding remarks and trends for future work.

2. Stochastic binary systems and separable systems

In this paper we will use the following definitions and notation.

Definition 1 (*SBS*). A *stochastic binary system* (SBS) is a triad $\mathcal{S} = (S, r, \phi)$:

- S: ground (finite) set of *components*, usually $S = \{1, \dots, N\}$; a configuration or a *state* of the system is an element of $\Omega = \{\sigma : S \to \{0,1\}\}$.
- r: probability measure on Ω.
- $\phi : \Omega \to \{0,1\}$: *structure* function.

Given a state $\sigma \in \Omega$, $\phi(\sigma) = 1$ means that the system is in an operational state; we call σ a *pathset*. Respectively, if $\phi(\sigma) = 0$ then the system is in a failure state; we call σ a *cutset*.

The reliability of a SBS is its probability of correct operation:

$$R_{\mathcal{S}} = P(\phi = 1) = \sum_{\sigma \in \Omega : \phi(\sigma)=1} r(\sigma). \tag{1}$$

The *unreliability* of $\mathcal{S}$ is $U_{\mathcal{S}} = 1 - R_{\mathcal{S}}$.

Two special configurations are the all-operational state, $\mathbf{1} = (1, 1, \dots, 1)$, and the all-failed state, $\mathbf{0} = (0, 0, \dots, 0)$.

We will denote by $<$ the usual partial order in Ω, where $\omega < \sigma$ iff $\omega_i \le \sigma_i$ for all $i \in S$ and there exists $j \in S$ such that $\omega_j < \sigma_j$.

Definition 2 (*SMBS*). An SBS is *monotone* if the structure function ϕ is monotonically increasing with respect to the usual partial order in Ω (i.e, if $\omega < \sigma$ then $\phi(\omega) \le \phi(\sigma)$), $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = 1$. We denote such an SBS as a Stochastic Monotone Binary System (SMBS).

Definition 3 (*Minpaths/Mincuts*). Let $\mathcal{S} = (S, r, \phi)$ be an SMBS:

- A pathset σ is a *minpath* if $\phi(\omega) = 0$ for all $\omega < \sigma$.
- A cutset ω is a *mincut* if $\phi(\sigma) = 1$ for all $\sigma > \omega$.
- A σ-ray is the set $S_\sigma = \{\omega \in \Omega : \omega \ge \sigma\}$.

An SMBS is fully characterized by its mincuts (or its minpaths). In fact, if we are given the complete list of minpaths, then the complete list of pathsets is precisely the union of the σ-rays among all minpaths σ.

As the class of SMBSs include the classical K-terminal graph reliability problem, which is known to belong to the $\mathcal{NP}$-Hard class [8], the reliability evaluation of an SMBS belongs to the class of $\mathcal{NP}$-Hard problems. Of course the same applies for the (still more general) problem of reliability evaluation of an SBS.

We consider in what follows $S = \{1, \dots, N\}$, so that $\Omega = \{0,1\}^N$ is the set of the extremal points of the unit hypercube in $\mathbb{R}^N$.

A hyperplane in the Euclidean space $\mathbb{R}^N$ is fully characterized by its normal vector n and a point P that belongs to the hyperplane: $\langle n, X - P \rangle = 0$, where $\langle x, y \rangle = \sum_{i=1}^N x_i y_i$ is the inner product. If we denote $n = (n_1, \dots, n_N)$ and $\langle n, P \rangle = \alpha_0$, the points of the hyperplane are those satisfying the equation $\sum_{i=1}^N n_i x_i = \alpha_0$. For ease of discussion (and without losing generality) we will choose the representation of a hyperplane so that any cutset ω lies on the hyperplane or in its negative side (i.e. the geometric points that verify $\sum_{i=1}^N n_i \omega_i \le \alpha_0$), and any pathset σ lies on the positive side of the hyperplane (i.e. $\sum_{i=1}^N n_i \sigma_i > \alpha_0$).

Such representation is justified by the following observation: for any separating hyperplane H, there exists $H' \sim H$ (that is, a hyperplane H' separating the same subset of cutsets and pathsets as H), with non-negative components of the normal vector, such that $\|n\|_1 = \sum_{i=1}^N n_i = 1$.

Definition 4 (*Separable System*). An SBS is *separable* if the cutsets/pathsets can be separated by some hyperplane in $\mathbb{R}^N$.

Fig. 1 shows an example of a separable SBS with three components. A generic state $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is operative if and only if at least two components satisfy $\sigma_i = 1$. In that example a separating hyperplane is defined by the equation $\sum_{i=1}^3 n_i x_i = \frac{1}{3}$, where the normal vector is $n = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

It turns out that separable systems are a special sub-class of SMBSs. However, in Section 3 we will build an infinite family of examples of SMBSs that are not separable.

3. Analysis of separable systems

In this section we present results concerning the complexity and characterization of separable systems. Even though separable systems accept an efficient representation, their reliability evaluation is computationally hard.

Theorem 1 (*Theor. 4, [5]*). *The reliability evaluation of separable systems belongs to the class of $\mathcal{NP}$-Hard problems.*

Separable systems can be characterized using Hahn–Banach separation theorem for compact and convex sets [9]. If $CH(\mathcal{P})$ and $CH(\mathcal{C})$ denote the convex hull of the pathsets and cutsets respectively, then following result holds:

Theorem 2 (*Prop. 3, [7]*). *An SBS is separable iff $CH(\mathcal{P}) \cap CH(\mathcal{C}) = \emptyset$.*

While the structure function of some SMBS can be defined by a hyperplane, there exist SMBSs that are not separable. In fact, consider an arbitrary number of components $N \ge 4$, and the SMBSs family $\mathcal{S}_N$ characterized by two mincuts $\{\omega^1, \omega^2\}$ such that $\omega^2 = \mathbf{1} - \omega^1$, and ω^1 is defined as follows:

$$\omega^1_i = 1,\ \forall i = 1, \dots, \lfloor N/2 \rfloor,$$
$$\omega^1_i = 0,\ \forall i = \lfloor N/2 \rfloor + 1, \dots, N.$$

Consider the states σ^1 and σ^2, such that $\sigma^2 = \mathbf{1} - \sigma^1$, and

$$\sigma^1_{2i-1} = 1,\ \forall i = 1, \dots, \lfloor N/2 \rfloor,$$
$$\sigma^1_{2i} = 0,\ \forall i = 1, \dots, \lfloor N/2 \rfloor.$$

First, observe that σ^1 and σ^2 are not upper-bounded by ω^1 nor ω^2 (i.e., the inequalities $\sigma^i \le \omega^j$ do not hold, for any pair $i, j \in \{1, 2\}$). Therefore, σ^1 and σ^2 must be pathsets, since $\mathcal{S}_N$ is the SBS characterized

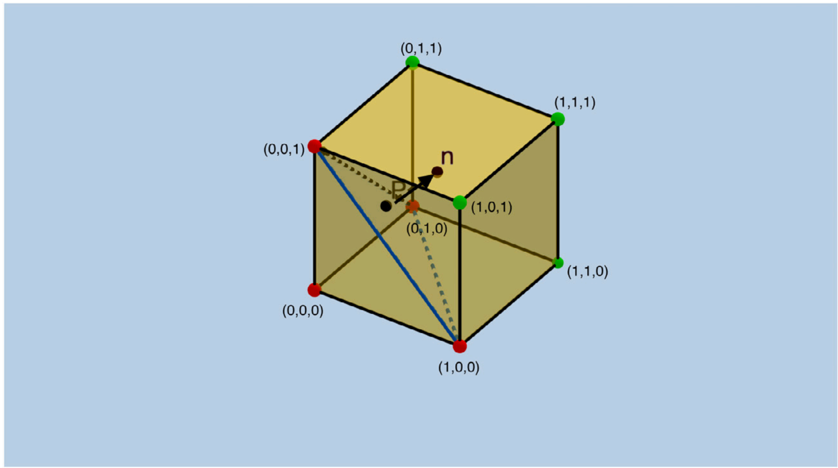

Fig. 1. States of the SBS are represented by the vertices of the cube. Blue and dashed lines correspond to the intersection of the separating plane (containing a point P, and with normal vector n) with the lateral faces of the cube. Red vertices are the non-operative states, and green vertices are the operative ones. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

by the mincuts ω^1 and ω^2. Further, $\frac{\sigma^1+\sigma^2}{2} = \frac{\omega^1+\omega^2}{2} = \frac{1}{2}\mathbf{1}$. By Hahn–Banach theorem, the infinite family of SMBSs $\{S_N\}_{N\geq 4}$ is not separable (see Theorem 2). By an exhaustive analysis, it can be observed that all SMBS are separable systems when the number of components is not greater than three. A geometric interpretation is also feasible in this cases. However, the challenges arise in higher dimensions.

In the following, we consider an alternative characterization of separable systems in terms of weighted cutsets and pathsets. Consider an arbitrary assignment $n_1,\dots,n_N$ of non-negative numbers to the respective components of the system. The condition $\sum_{i:\sigma_i=1} n_i \geq \alpha_0$ for all the pathsets is equivalent to finding the pathset σ with minimum-cost, $c(\sigma) = \sum_{i:\sigma_i=1} n_i$, and testing if $c(\sigma) \geq \alpha_0$. Analogously, the condition $\sum_{i:\omega_i=1} n_i < \alpha_0$ for all the cutsets is equivalent to testing whether the cutset ω with minimum cost, $c(\omega) = \sum_{i:\omega_i=0} n_i$, satisfies the test $S - c(\omega) < \alpha_0$, where $S = \sum_{i=1}^N n_i$ is the cost of the global system. Observe that, for convenience, the cost of a cutset is defined as the sum of the components under failure. In particular, we get the following characterization of separable systems:

Theorem 3. *An SBS is separable if and only if there exists an assignment of non-negative costs to the components $\{n_i\}_{i=1,\dots,N}$ such that $S < c(\sigma) + c(\omega)$, where $c(\sigma)$ and $c(\omega)$ denote pathset/cutset minimum-cost respectively.*

Proof. First, let us assume that we have a separable SBS with hyperplane $\sum_{i=1}^N n_i x_i = \alpha_0$. Using the previous reasoning, the assignment $\{n_i\}_{i=1,\dots,N}$ verifies $c(\sigma) \geq \alpha_0$ and $S - c(\omega) < \alpha_0$. Therefore, $S < c(\omega) + c(\sigma)$.

For the converse, let us fix $\alpha_0 = c(\sigma)$, the pathset with minimum cost. Clearly, the specific pathset σ meets the condition $\sum_{i=1}^N n_i\sigma_i \geq \alpha_0$; in fact the equality is met. By its definition, the inequality holds for the other pathsets. Finally, we use the fact that $S < c(\omega)+c(\sigma)$ to verify that the cutset with minimum-cost, ω, meets the inequality $\sum_{i=1}^N n_i\omega_i < \alpha_0$. The inequality for the other cutsets is straight since ω is a cutset with minimum-cost. Therefore, the SBS is separable, concluding the proof. $\square$

4. Separability in graphs

Our characterization of separable systems has a straightforward reading in the all-terminal reliability model.

Definition 5 (*Separable Graph*). A graph $G = (V, E)$ is *separable* if there exists an assignment of non-negative real numbers $n_1,\dots,n_m$ to its m links, and there exists a threshold α such that $c(E') \geq \alpha$ if and only if the spanning subgraph $G' = (V, E')$ is connected.

Let G be a connected weighted graph. Recall the Kruskal algorithm provides efficiently the cost of the minimum spanning tree, $MST(G)$. Furthermore, the cutset with minimum-cost, $m(G)$, is obtained using the Ford–Fulkerson algorithm. Therefore, the following corollary of Theorem 3 holds for graphs:

Corollary 1. *A graph is separable iff there exists a feasible assignment $\{n_i\}_{i=1,\dots,N}$ to the links such that $S < MST(G) + m(G)$, where $MST(G)$ is the cost of the minimum spanning tree, $m(G)$ the mincut with minimum capacity, and $S = \sum_{i=1}^N n_i$ the sum of the link weights.*

For example, trees and elementary cycles are separable graphs. Indeed, if T_n is a tree with n nodes, a feasible assignment is an identical unit-cost to all the links, since in that case $MST(T_n) = n-1$, $m(G) = 1$ and the global sum is $S = n-1 < MST(T_n) + m(T_n)$. Analogously, if C_n denotes the elementary cycle with n nodes, then $S = n < (n-1)+2 = MST(C_n) + m(C_n)$, and the same unit-cost assignment works.

Intuitively, if the graph is dense enough, one would expect that the combined cost of a minimum spanning tree and mincut should not exceed S, the global cost of the graph.

Our first result deals with the extremal case of complete graphs:

Proposition 1. *Complete graphs $(K_n)_{n\geq 4}$ are nonseparable.*

Proof. Consider an arbitrary assignment $\{n_i\}_{i=1,\dots,n(n-1)/2}$ to the links of K_n, and an arbitrary star-graph $K_{1,n}$ contained in K_n. Since $K_{1,n}$ is connected, its cost is greater than, or equal to the cost of the minimum spanning tree, so, $c(K_{1,n}) \geq MST(K_n)$. Furthermore, the complementary links of $K_{1,n}$, or the complementary graph $K_{1,n}^C$, is a cutset (it isolates a single node), so the cost must exceed the mincut: $c(K_{1,n}^C) \geq m(K_n)$. But then, the global cost is $c(K_n) = c(K_{1,n}) + c(K_{1,n}^C) \geq MST(K_n)+m(K_n)$. The conclusion is that $S = c(K_n) \geq MST(K_n)+m(K_n)$ for any feasible assignment, and K_n is nonseparable. $\square$

With the following lemmas, we will present a hereditary property of separable graphs, stated in Theorem 4. Consider a simple connected graph $G = (V, E)$. We will consider two different link additions:

- We denote $G_{in} = G + e_{in}$ to the resulting graph after the addition of an internal link $e_{in} = \{u_1, u_2\}$, where $u_1, u_2 \in V$.
- We denote $G_{out} = G + e_{out}$ to the resulting graph after the addition of an external link $e_{out} = \{u_1, u_2\}$, where $u_1 \in V$ but $u_2 \notin V$.

Observe that $G + e_{in}$ and G share an identical node-set V, while the node-set for $G + e_{out}$ is $V \cup \{u_2\}$.

Lemma 1. *If G is nonseparable then G_{out} is nonseparable.*

Proof. Suppose for a moment that there exists a feasible assignment $\{n_i\}_{i=1,\dots,N+1}$ for G_{out}. Then:

$$\begin{aligned}(\sum_{i=1}^{N} n_i) + n_{N+1} &< MST(G_{out}) + m(G_{out}) \\ &= MST(G) + n_{N+1} + \min\{m(G), n_{N+1}\} \\ &\leq MST(G) + n_{N+1} + m(G),\end{aligned}$$

and $\{n_i\}_{i=1,\dots,N}$ would be a feasible assignment for G, which is a contradiction. Therefore, G_{out} is nonseparable. □

Lemma 2. *If G is nonseparable then G_{in} is nonseparable.*

Proof. Suppose for a moment that there exists a feasible assignment $\{n_i\}_{i=1,\dots,N+1}$ for G_{in}. Then:

$$\begin{aligned}(\sum_{i=1}^{N} n_i) + n_{N+1} &< MST(G_{in}) + m(G_{in}) \\ &\leq MST(G) + m(G) + n_{N+1},\end{aligned}$$

and $\{n_i\}_{i=1,\dots,N}$ would be a feasible assignment for G, which is a contradiction. Therefore, G_{in} is nonseparable. □

Observe that Lemma 2 informally states that graphs with more density are nonseparable. Using the contrapositive of Lemmas 1 and 2 we obtain the following:

Theorem 4. *Separability is a hereditary property in graphs.*

Proof. Reading the contrapositive of Lemma 2, we know that the deletion of one or several links from a separable graph is also separable. By Lemma 1, we also know that a node-deletion in a separable graph (with the intermediate deletion of links using Lemma 2) is also separable. Combining node and link deletions, an arbitrary subgraph is obtained, and it must be separable as well. □

Lemma 3. *If G is separable, G_{out} is also separable.*

Proof. Consider a feasible assignment $\{n_i\}_{i=1,\dots,N}$ for G, where $S < MST(G) + m(G)$ holds. Let us consider an extended assignment with n_{N+1} for the external link, such that $n_{N+1} > m(G)$. Then:

$$\begin{aligned}S + n_{N+1} &< (MST(G) + n_{N+1}) + m(G) \\ &= MST(G_{out}) + \min\{m(G), n_{N+1}\} \\ &= MST(G_{out}) + m(G_{out}),\end{aligned}$$

and $\{n_i\}_{i=1,\dots,N+1}$ is a feasible assignment for G_{out}. □

We define a cycle with arborescences as a connected graph with a single cycle. In a cycle with arborescences, each node either belongs to the single cycle, or belongs to a tree "dangling" from a node in the cycle.

It is interesting to observe that any connected graph with the same number of nodes and links is either a cycle or a cycle with arborescences.

Corollary 2. *Cycles with arborescences are separable graphs*

Proof. We know that elementary cycles are separable. The result follows by the addition of one or several trees hanging to different nodes from the first cycle. Supported by Lemma 3, the separability is preserved by the addition of those links. □

Fig. 2 depicts Monma graphs. These graphs have two degree-3 nodes connected by 3 node-disjoint paths. Every proper subgraph of a Monma graph is either a unicyclic graph, a tree, or a disconnected graph. Therefore, every proper subgraph of a Monma graph is separable. We will see that Monma graphs are minimally nonseparable graphs. Clyde Monma et al. used these graphs to design minimum cost biconnected metric networks [10]. Some (but not all) of these graphs also attain the maximum reliability among all the graphs with p nodes and $q = p + 1$ links [11].

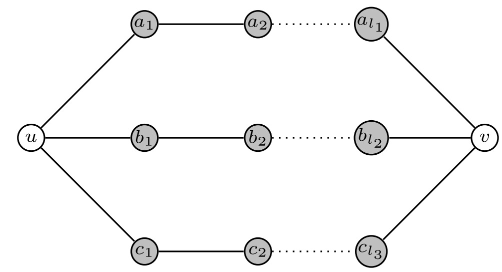

Fig. 2. Monma graph M_{l_1+1,l_2+1,l_3+1}.

Lemma 4 (*L. Stábile*)**.** *Monma graphs are nonseparable*

Proof. Consider an arbitrary order for the links of Monma graph, and the rule $\phi(\sigma) = 1$ iff the Monma subgraph given by the links in subgraph σ is connected. We will show that the convex hull of pathsets and cutsets meet at some point, and the result is established by Theorem 2. Consider the four links $e_1 = \{u, a_1\}$, $e_2 = \{a_1, a_2\}$, $e_3 = \{u, b_1\}$ and $e_4 = \{b_1, b_2\}$ from Fig. 2. Let $1_{e_i,e_j}$ denote the binary word that is set to 1 in all the bits but 0 in the positions corresponding to the links e_i and e_j. Consider the following identity:

$$\frac{1}{2}(1_{e_1,e_2} + 1_{e_3,e_4}) = \frac{1}{4}(1_{e_1,e_3} + 1_{e_1,e_4} + 1_{e_2,e_3} + 1_{e_2,e_4}) \quad (2)$$

On one hand, we have a convex combination of cutsets. On the other, a convex combination of pathsets. By Theorem 2, Monma graphs are nonseparable. □

Recall that a node v in a graph G is a cut-point if $G - v$ has more components than G. A connected graph is biconnected if it has no cut-points. The addition of an ear in a graph G is the addition of an external elementary path between two different nodes from G. Whitney characterization theorem for biconnected graphs asserts that there exists an ear decomposition of all biconnected graphs, such that $G = C_s \cup H_1 \cup H_2 \cup \cdots \cup H_r$, C_s is an elementary cycle and H_i is the addition of an ear to the previous graph [12]. A proof using modern terminology is given in the classical graph theoretical book [13]. This structural characterization of biconnected graphs leads us immediately to the following:

Theorem 5. *Biconnected graphs are nonseparable, except for elementary cycles.*

Proof. As the base-step, we know by Lemma 4 that Monma graphs are nonseparable. If G is biconnected and it is not an elementary cycle, then it has the addition of at least one ear of a cycle. Therefore, it has Monma as a subgraph. Therefore, Theorem 4 asserts that G cannot be separable. □

Recall that the link-connectivity of a graph G is the least number of links that must be removed in order to disconnect G. The Butterfly-graph consists of two triangles meeting in a common point (see Fig. 3). This is the smallest graph with link connectivity 2 that is not biconnected, since the kissing-point is a cut-point. As a consequence, it is natural to decide the separability of this graph:

Lemma 5. *The Butterfly-graph B is nonseparable*

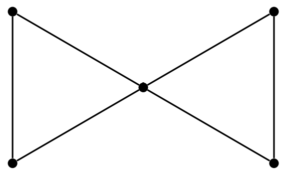

Fig. 3. Butterfly-graph B.

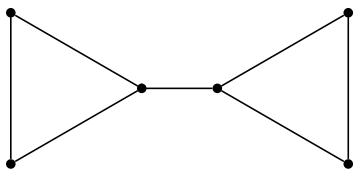

Fig. 4. Glasses-graph B_e.

Proof. Consider an arbitrary assignment $\{n_i\}_{i=1,\dots,6}$ for the links. We consider an assignment $n_1 \leq n_2 \leq n_3$ in the left triangle, and $n_4 \leq n_5 \leq n_6$ in the right triangle. Therefore $MST(B) = S - n_3 - n_6$, and $m(B) = \min\{n_1+n_2, n_4+n_5\} \leq \min\{2n_2, 2n_5\} \leq n_3+n_6$. This implies that $MST(B) + m(B) \leq S$ for all possible assignments in B, and B has no feasible assignment. □

An analogous reasoning leads to the following generalization:

Corollary 3. *Two kissing cycles are nonseparable.*

A further generalization recalls Whitney characterization for bridgeless graphs: G is a bridgeless graph if and only if $G = C_s \cup H_1 \cup H_2 \cup \dots \cup H_r$, C_s is an elementary cycle and H_i is the addition of an ear or a kissing cycle to the previous graph. The following result is analogous to Theorem 5:

Corollary 4. *Bridgeless graphs are nonseparable, except for elementary cycles.*

In order to fully characterize separable graphs, we need to study graphs that have at least one bridge $e \in G$. We already know that all the links in a tree are bridges, and they are separable graphs. Furthermore, cycles with arborescences are separable as well. Let us proceed our analysis with two triangles linked by a single bridge e, a graph called the Glasses-graph B_e (see Fig. 4).

Lemma 6. *The Glasses-graph B_e is nonseparable.*

Proof. The reasoning is identical to the Butterfly-graph. Consider an assignment $\{n_i\}_{i=1,\dots,7}$ as in the Butterfly-graph, but n_7 is the assignment for the bridge e. Therefore:

$$\begin{aligned} MST(B_e) + m(B_e) &= (S - n_3 - n_6) \\ &\quad + \min\{n_1+n_2, n_4+n_5, n_7\} \\ &\leq S, \end{aligned}$$

since $\min\{n_1+n_2, n_4+n_5, n_7\} \leq n_3+n_6$, and the last inequality was already proved for the Butterfly-graph. □

A slight generalization is possible:

Corollary 5. *Two cycles linked by an elementary path are nonseparable.*

We are now ready to fully characterize separable graphs:

Theorem 6. *A graph G is separable iff G falls into one of the four categories:*

1. *G is not connected;*
2. *G is a tree;*
3. *G is an elementary cycle;*
4. *G is an elementary cycle with arborescences.*

Proof. The proof of the reverse direction is easy, as all graphs in the four categories are separable:

1. If G is disconnected, all of its configurations are cutsets and the reliability is null. In this case, the inequality $\sum_{i=1}^N \sigma_i > 2N$ is not satisfied by any binary vector $\sigma = (\sigma_1, \dots, \sigma_N)$, and the graph is separable.
2. If G is a tree T_N with N links, the evidence is the hyperplane $\sum_{i=1}^N \sigma_i \geq N$.
3. If $G = C_N$ is an elementary cycle, the evidence is the inequality $\sum_{i=1}^N \sigma_i \geq N-1$.
4. If G is a tree with arborescences, Corollary 2 states that G is separable.

To prove the direct direction, let G be a separable graph, and assume G is connected. We know by Corollary 4 that G must have a bridge. Combining Theorems 4 and 5, we know that every subgraph of G must be an elementary cycle. Combining Corollaries 3 and 5, G cannot have two cycles (either they are kissing or connected by a path). Therefore, G is either a tree, an elementary cycle or an elementary cycle with arborescences. □

The complexity of determining if a graph is separable under the all-terminal reliability model is linear. It is enough to employ an algorithm for finding if the graph is connected (for instance, a Depth First Search or Breadth First Search, both of linear complexity). If it is not connected, it is separable. If it is connected, and the number of edges is less or equal to the number of nodes, then it is either a tree, a cycle or an elementary cycle with arborescences, and it is separable. If it is connected and the number of edges is strictly larger than the number of nodes, then it is not separable.

Corollary 6. *The all-terminal reliability evaluation of separable graphs (in the case of independent elementary reliabilities) belongs to the class $\mathcal{P}$ of polynomial-time problems.*

Proof. The analysis is straightforward. Let G be a separable graph:

1. If G is not connected, then $R(G) = 0$.
2. If $G = T_N$ is a tree with N links with independent reliabilities $(p_e)_{e \in T_N}$, then $R(G) = \prod_{e \in T_N} p_e$.
3. If $G = C_N$, then
$$R(C_N) = \prod_{e \in C_N} p_e + \sum_{e \in C_N} (1-p_e) \prod_{e' \neq e} p_{e'}.$$
4. Finally, if G is an elementary cycle with arborescences: $G = C_l \cup T_s$, being T_s union of trees pending from the cycle C_l. Therefore, $R(G) = R(C_l) \times \prod_{e \in T_s} p_e$.

The reader can appreciate that the reliability computation is a product, or a sum of products of the elementary link reliabilities. Therefore, the number of operations involved are linear, or quadratic, in the number of links. □

The corank of a graph is the number of independent cycles. In a connected graph with n nodes and m links, its corank is precisely $c(G) = m - n + 1$. It is worth to remark that Theorem 6 can be re-stated in terms of corank: a connected graph G is separable if and only if its corank is either 0 or 1.

We close this section by discussing a connection between the combinatorial optimization problem called the Network Utility Problem (NUP) and separable graphs. First, observe that an arbitrary spanning tree of a connected graph G has $n-1$ links. Therefore, the corank of a graph is precisely the number of *additional links that we must pay* to build the graph G, starting from a minimally-connected graph. In terms of communication, the corank of G represents *redundancy*. At the cost of redundancy, the resulting network can be robust under a certain amount of link failures. The profit is the link connectivity $\lambda(G)$, which represents the lowest number of links that should be removed in order to disconnect G. As a consequence, the *utility* of a graph, $u(G)$, is the difference between the connectivity and the corank: $u(G) = \lambda(G) - c(G) = \lambda - m + n - 1$. In [14], the authors formally proved the following

Theorem 7. *The graphs with maximum utility are exactly the trees and cycles. Their utility value is* 1.

Corollary 7. *All the graphs with maximum utility are separable graphs.*

The all-terminal reliability polynomial under identical elementary reliabilities in the links r is

$$R_G(r) = \sum_{i=\lambda(G)}^{c(G)-1} n_i(G)p^{m-i}(1-p)^i + \tau(G)p^{n-1}(1-p)^{m-n+1}, \quad (3)$$

where $n_i(G)$ is the number of connected subgraphs of G with precisely $m-i$ links, and $\tau(G)$ is the tree-number of G, which can be found using Kirchhoff's Matrix-Tree theorem [15]. Therefore, the number of unknowns is precisely the number of terms involved in the summation: $c(G) - \lambda$. The only cases where there are no terms in the sum occur either when $c(G) - \lambda = -1$, exactly in trees and cycles, or when $c(G) - \lambda = 0$, only in an elementary cycle with arborescence, K_4, the Kite-graph and the Butterfly-graph [14]. These graphs are considered as the simplest in terms of reliability analysis. Indeed, in [14] the authors define the *level of difficulty* of a graph as the difference $d(G) = c(G) - \lambda - 1$, and a graph is easy if and only if $d(G) \leq 0$:

Corollary 8. *All separable graphs are easy graphs.*

The reader can observe that the graphs with maximum utility $u(G)$ are the easiest graphs, with the minimum level of difficulty $d(G)$.

5. *d*-separability

A natural extension of our prior analysis is a classification of non-separable systems.

Let $\mathcal{S} = (S, r, \phi)$ be an arbitrary SMBS, and consider its corresponding 0-1 labels of the vertices of a hypercube Q_N in the Euclidean space $\mathbb{R}^N$.

Definition 6 (*Level of Separability*)**.** The *level of separability of* $\mathcal{S}$ is the least positive integer d such that occurs one of the following conditions:

- there exist d hyperplanes such that all the pathsets reside in the intersection of the d–half spaces specified by the non-negative normal components of the hyperplanes; or
- there exist d hyperplanes such that all the cutsets reside in the intersection of the d–half spaces specified by the opposite of the non-negative normal components.

Proposition 2. *Let $\mathcal{S}$ be an arbitrary SMBS, and let $\mu = |\mathcal{MC}|$ be the number of all its mincuts of $\mathcal{S}$. Then the level of separability d is at most μ.*

Proof. Suppose that $\omega^1, \dots, \omega^\mu$ is the list of all the mincuts of $\mathcal{S}$. Consider the sets $S_i = \{j : \omega_j^i = 0\}$, that represent the non-operational states for the mincut ω^i. Observe that the mincut ω^i does not meet the inequality $\pi_i : \sum_{j \in S_i} \omega_j^i \geq 1$. Furthermore, the hyperplanes $\pi_1, \dots, \pi_\mu$ meet Definition 6, and the result follows. □

Proposition 2 shows that the level of separability will always be well defined for any arbitrary SMBS, thus it is an alternative way to classify a notion of *difficulty* in the reliability evaluation for SMBSs.

If we return to the all-terminal reliability model, we know all the graphs with level of separability $d = 1$ (i.e., all separable graphs). From Theorem 6 we can observe that the Butterfly-graph, the Glasses-graph and Monma represent minimally nonseparable cases. For a better understanding of Definition 6, we find the level of separability in these minimally nonseparable cases in the following paragraphs.

Let us denote x_1, x_2, x_3 and y_1, y_2, y_3 the states of the links for the Butterfly-graph, corresponding to both triangles (see Fig. 3). All pathsets must have at least two links from every triangle, and the following 2 hyperplanes determine pathsets:

$$x_1 + x_2 + x_3 \geq 2$$
$$y_1 + y_2 + y_3 \geq 2.$$

Since we know that the Butterfly-graph is nonseparable, $d > 1$, and since the previous hyperplanes fulfill the definition, the level of separability for the Butterfly-graph is $d = 2$.

Analogously, if we link both triangles with a new link z, we get the Glasses-graph. A slight modification of the hyperplanes shows that the Glasses-graph has level of separability $d = 2$:

$$x_1 + x_2 + x_3 + 3z \geq 5$$
$$y_1 + y_2 + y_3 \geq 2.$$

Observe that we *force* the link z to be operational, adding the term $3z$ in the first hyperplane. Finally, let us consider Monma graph $M_{2,2,1}$ from Fig. 2, where the three paths have respective lengths 2, 2 and 1, and the respective links from each path are sequentially identified with the binary states x_1, x_2, y_1, y_2 and z. The reader is invited to check that the level of separability in Monma graph $M_{2,2,1}$ is also $d = 2$, and the following pair of hyperplanes works:

$$10x_1 + 10x_2 + y_1 + y_2 + z \geq 12$$
$$x_1 + x_2 + 10y_1 + 10y_2 + z \geq 12.$$

Currently, there is no constructive algorithm to produce the minimum number of hyperplanes for an SMBS. We wish to develop a complementary theory to the one presented in Section 4 for separable graphs, but finding the correct level of separability for any given graph. Inspired by Corollary 6, we propose the following:

Conjecture 1. *Let d be a fixed positive integer. Then, the all-terminal reliability evaluation of graphs with level of separability d (under the hypothesis of independent elementary reliabilities) belongs to the class $\mathcal{P}$ of polynomial-time problems.*

6. Conclusions

In this work, we study the reliability evaluation of stochastic binary systems (SBS) and some properties arising from the definition of separability, and we apply these concepts to take a new look at the all-terminal reliability model.

An efficient representation of separable systems is presented, and a full characterization of these special systems is obtained for some particular models. The major strength of separable systems is their efficient representation. The major shortcoming is that the reliability evaluation is still $\mathcal{NP}$-hard.

Separable systems accept polynomial-time reliability evaluation when restricted to the all-terminal reliability model. This result was discovered using functional analysis and feasible functionals from the links of a graph, meeting separability constraints.

As future work, we would like to establish Conjecture 1 for a better understanding of nonseparable systems, and the interplay between general SBSs and the all-terminal reliability model, which has a wide spectrum of applications.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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