Universal Reliability Bounds for Sparse Networks

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***Abstract*—Consider a graph with perfect nodes and edges subject to independent random failures with identical probability. The *all-terminal reliability* (ATR) is the probability that the resulting subgraph is connected. First, we fully characterize uniformly least reliable graphs (ULRG) whose co-rank is not greater than four. Universal reliability bounds are here introduced for those graphs. It is formally proved that ULRG are invariant under bridge-contractions, and maximize the number of bridges among all connected simple graphs with a prescribed number of nodes and edges. A closed-form for the maximum number of bridges is also given, which has an intrinsic interest from a graph-theoretic point of view. Finally, the cost-reliability trade-off is discussed, comparing the number of edges required to reduce the reliability gaps between the least and most reliable graphs. A remarkable conclusion is that the network design is critical under rare event failures, where the reliability-gap between least and most-reliable networks is monotonically increasing with the number of terminals.**

***Index Terms*—All-Terminal Reliability, Reliability Bounds, Uniformly Most Reliable Graphs, Uniformly Least Reliable Graphs.**

ACRONYMS[1]

ULRG	Uniformly Least Reliable Graph
UMRG	Uniformly Most Reliable Graph
ATR	All-Terminal Reliability
RVR	Recursive Variance Reduction
IS	Importance Sampling

NOTATIONS

G	A connected graph representing the network
R_G	All-terminal Reliability of G
U_G	Unreliability of G
n	Number of nodes in G
e	Number of edges in G
$\Omega(n, e)$	Collection of connected, simple undirected graphs with n nodes and e edges
i	$= e - n + 1$ co-rank of $G \in \Omega(n, e)$
ρ	Failure probability of the edges
$f_k(G)$	Number of k-operational subgraphs of G
$m_k(G)$	Number of k-cuts for a given graph G
$\lambda(G)$	Edge-Connectivity of a graph G
$\tau(G)$	Number of Spanning-trees, or tree-number of G
$b(n, e)$	Maximum number of bridges of a graph $G \in \Omega(n, e)$

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[1]The singular and plural of an acronym are always spelled the same.

I. INTRODUCTION

SEVERAL metrics for network reliability analysis have been proposed, when the system can be represented by some graph $G = (V, E)$ subject to random failures on its components (i.e., nodes and edges). The reader is invited to consult an excellent monograph on the combinatorics of network reliability authored by Colbourn [1]. Given its paramount importance, the all-terminal reliability (ATR) has a large body of related work. This metric is the connectedness probability of a random graph, subject to independent edge failures with identical probability. Provan and Ball formally proved that the ATR evaluation belongs to the class of $\mathcal{NP}$-Hard problems [2]. Furthermore, it is not known whether this problem belongs to the $\mathcal{NP}$ class or not, an enigmatic question that attracts theoretical computer scientists as well [3].

Given the hardness of the exact reliability evaluation, approximative methods were developed, as well as reliability bounds. Several non-exact methods are based on Monte-Carlo simulation, whose most basic setting considers an averaging of independent identically-distributed random networks. The classical book authored by Fishman [4] contains applications to network reliability analysis as well as other fields. An outstanding method from this class is Recursive Variance Reduction (RVR), which successively reduces the network size using iterative conditional measures [5]–[7]. Importance Sampling (IS) is based on Radon-Nikodym change of measure [8]. A sequence of approximate zero variance IS for the network reliability estimation of highly-reliable systems is given in [9]. Dagger Sampling [10] and Cross-Entropy [11] were also proposed; the interested reader can find a good point of departure for approximative methods in the previous works and references therein. A thorough treatment of Rare Event simulation is covered in the book [12] for the analysis of highly reliable systems, where the unreliability represents a rare event.

Exact methods are also available, with exponential-time complexity for general instances. An exhaustive probability-sum among all the operational configurations is valid, but it does not take into account possible factorizations. An alternative is to consider sums of disjoint products (SDP), where the goal is to group terms of a logical function that represents operational states (the atom x_e in this logical function is the states of the edge e, and $f(\{x_e\}_{e \in E}) = 1$ iff the resulting graph is connected). Another approach is to observe that an operational state always includes a minimally operational state. Therefore, we can list all the minimally operational states and adopt the Inclusion-Exclusion principle for the union among all the minimally operational states, called *minpaths* in general, or *trees* in the special ATR setting.

The previous exact approaches have clear shortcomings, and they work only for special networks due to computational feasibility [13]. A factorization theory is already mature. For instance, series-parallel graphs accept a linear-time reliability evaluation [14]. The main tool is to condition over the state of a single link, and apply deletion-contraction formula [15]. This concept is available from the final half of the previous century. However, the best selection of this pivotal edge is a pioneer work from Satyanarayana et al. [16]. In [14], it is formally proved that all series-parallel graphs accept a linear-time K-terminal reliability evaluation, using the concept of polygon-to-chain reductions. However, most real-life networks are not series-parallel. It is worth while to remark that the applicability of exact methods is limited, and valid for small or medium-sized networks, series-parallel networks, or recursively defined networks with special symmetry [17].

The literature also offers reliability bounds, which are based on dropping terms in the exact methods [18], or counting techniques [19]. However, the effort is devoted to finding reliability bounds for *specific* graphs, and not *universal reliability bounds* for all the graphs with a prescribed number of nodes and edges.

Throughout the document we will consider, without loss of generality, connected graphs, since the ATR of a disconnected graph is zero. Denote $R_G(\rho)$ the ATR for the graph G, whose edges are subject to random failures with identical probability ρ. Let $\Omega(n,e)$ be the collection of connected simple graphs with n nodes and e edges. A lower reliability bound $l(\rho)$ is *universal* for the collection $\Omega(n,e)$ if $l(\rho) \leq R_G(\rho)$, for all $G \in \Omega(n,e)$ and all $\rho \in [0,1]$. Analogously, an upper reliability bound $u(\rho)$ is *universal* for $\Omega(n,e)$ if $R_G(\rho) \leq u(\rho)$ for all $G \in \Omega(n,e)$ and all $\rho \in [0,1]$:

Definition 1. *A pair of functions* $[l(\rho),u(\rho)]$ *is a* universal bound *for* $\Omega(n,e)$ *if* $l(\rho) \leq R_G(\rho) \leq u(\rho)$ *for all* $G \in \Omega(n,e)$ *and all* $\rho \in [0,1]$.

Definition 2. *The best universal bounds* $[l_b(\rho),u_b(\rho)]$ *meet the inequalities* $l(\rho) \leq l_b(\rho)$ *and* $u_b(\rho) \leq u(\rho)$*, for any given universal bound* $[l(\rho),u(\rho)]$.

Here we also consider *uniformly least-reliable graphs* (ULRG), whose reliability is the least among a graph-set with a prescribed number of nodes and edges.

The contributions of this work can be summarized by the following items:

1) It is formally proved that ULRG maximize the number of bridges $b(n,e)$ in $\Omega(n,e)$. A closed-form for $b(n,e)$ is given, which has intrinsic interest from a graph-theoretic viewpoint.
2) It is formally proved that ULRG are invariant under bridge-contractions. This invariance property is in consonance with a conjecture posed in 1990 by Boesch et al. on the characterization of ULRG [20]. A stronger conjecture is here proposed, which implies the former.
3) The infinite sequence of ULRG for all the pairs (n,e) such that $e \leq n+3$ is fully characterized. The exact ATR evaluation is given for those infinite families of graphs.
4) The best universal bounds for all the networks such that $e \leq n+3$ are introduced.
5) Finally, the cost-reliability trade-off is discussed. A remarkable conclusion is that a smart network design is essential when failures represent a rare event. This effect is quantified both analytically and numerically.

As far as the author knows, this is the first time where closed-forms for the reliability evaluation of both ULRG and UMRG are given, for all (n,e) such that $e \leq n+3$. New evidence is also given to build a full set of ULRG candidates, for all the pairs (n,e).

This document is organized as follows. Section II presents general concepts from graph theory, as well as the list of uniformly most reliable graphs (UMRG) such that $e \leq n+3$. These graphs will be useful to build universal upper-bounds. A new set of uniformly least reliable graphs (ULRG) is introduced in Section III. An extended graph-set $\{G_{n,e} : e \geq n-1\}$ is conjectured to be ULRG, with new evidence that this conjecture is true. An exact reliability evaluation for all the previous graphs is performed in Section IV, together with a summary of the best universal bounds. A discussion of the cost-reliability trade-off is provided in Section V. Finally, Section VI presents concluding remarks and trends for future work.

II. Uniformly Most Reliable Graphs

In this section, first we revisit general concepts from graph theory. Then, a list of UMRG such that $e \leq n+3$ is presented.

A. Concepts

The graph-theoretic terminology that will be used throughout this document is here presented. The reader can consult the book authored by Harary for further details [21].

A graph is *connected* if every pair of nodes are mutually reachable. A graph is *simple* if it has no loops nor multiple edges. A graph is *undirected* if the edges have no direction. Given a graph $G=(V,E)$ and an edge $e \in E$, the graph $G-\{e\}$ has the same node-set V but edge-set $E-\{e\}$. The graph $G-\{v\}$ for some node $v \in V$ has node-set $V-\{v\}$ and edge-set $E' = E - \{(u,v) : u \in V\}$.

A *bridge* is an edge $e \in E$ such that $G-\{e\}$ is disconnected. A *cut-point* is a node $v \in V$ such that $G-\{v\}$ is disconnected. A graph is *biconnected* if it has no cut-points. A *block* of a graph is a maximally biconnected subgraph.

The neighbors of v is $\mathcal{N}(v) = \{w \in V : (v,w) \in E\}$; the degree of a node v is $d(v) = |\mathcal{N}(v)|$. A *chain* is a sequence of adjacent nodes $(v_0,v_1),(v_1,v_2),\ldots,(v_{r-1},v_r)$ such that $d(v_i)=2$ for all $i \in \{1,\ldots,r-1\}$, and $d(v_0)>2$, $d(v_r)>2$. A chain is *trivial* if it consists of a single edge. If we replace every chain by a trivial chain, we obtain a new graph that is called the *distillation* $D(G)$ of a graph G. An *elementary subdivision* of the edge e is the replacement of $e=(u,v)$ by two edges: $e_1=(u,x)$ and $e_2=(x,v)$, where x is a new node, $x \notin V$. An *edge-contraction*, denoted $G*e$, is the resulting graph after the identification of the nodes u and v where $e=(u,v)$, and the neighboring-nodes of the new node $z=u=v$ is $\mathcal{N}(z) = \mathcal{N}(u) \cup \mathcal{N}(v) - \{z\}$. Observe that the contraction $G*e$ could produce repeated edges, but $G*e$ is

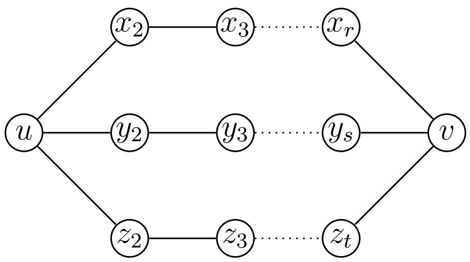

Fig. 1. θ-graph with lengths r, s and t, here denoted $\theta_{r,s,t}$

a simple graph when e is a bridge. Here we contract bridges only, so we are concerned with simple graphs.

The k-cuts are the edge-sets $E' \subseteq E$ such that $|E'| = k$ and the resulting graph $G - E'$ is disconnected. The number of k-cuts in a graph G is denoted as $m_k(G)$, or just m_k if there is no risk of confusion. Observe that $m_1(G)$ is the number of bridges. The least number λ such that $m_\lambda(G) > 0$ is the *edge-connectivity*, or connectivity of a graph. In a *complete graph* K_n, all the nodes are mutually adjacent. An elementary cycle C_n is a graph where all the nodes are configured in a cycle $v_1, v_2, \ldots, v_n, v_1$. A θ-graph is a graph which consists of three chains with identical end-points. Fig. 1 presents a θ-graph composed by three chains whose lengths are r, s and t, denoted as $\theta_{r,s,t}$. A *generalized θ-graph* $\theta_{l_1,l_2,\ldots,l_r}$ has r chains with identical endpoints, whose lengths are precisely $l_1, \ldots, l_r$ for some $r \geq 3$. A *star* $K_{1,n-1}$ is a graph where a central node is connected to all the others, and it has no additional edges. A *tree* is a connected graph whose edges are all bridges. Given a graph G, the *tree-number*, $\tau(G)$, is the number of spanning-trees. Note that $\tau(G) = \binom{e}{e-n+1} - m_{e-n+1}$, being $n = |V(G)|$ and $e = |E(G)|$ its respective number of nodes and edges. A connected graph G with n nodes and e edges has co-rank $i(G) = e - n + 1$. An open-ear is the addition of an external path $P_{u,v}$ that connects distinct nodes u and v in a graph G. A *matching* is a set of non-adjacent edges. A *perfect matching* is a matching that meets all the nodes of a graph.

B. UMRG such that $e \leq n + 3$

Our universe $\Omega(n, e)$ is the set of connected, undirected simple graphs with n nodes and e edges. If $G \in \Omega(n, e)$, the *reliability*, $R_G(\rho)$, is the probability that G is connected, where the edges fail independently, with probability ρ. For convenience, we sometimes deal with the *unreliability* $U_G(\rho) = 1 - R_G(\rho)$. The unreliability can be immediately obtained using sum-rule:

$$U_G(\rho) = \sum_{k=0}^{e} m_k(G)\rho^k(1-\rho)^{e-k}, \tag{1}$$

being $m_k(G)$ the number of k-cuts. Therefore, the unreliability evaluation is directly related with *counting*.

The concept of uniformly optimally-reliable graphs was introduced by Boesch, motivated by the design of highly-reliable networks [22]. Years later, Myrvold et al. replaced the previous term by uniformly most-reliable graphs (UMRG) to avoid a tongue-twister [23]:

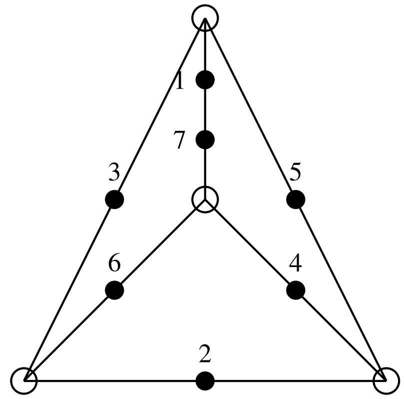

Fig. 2. Graph E_n for $n = 11$, after 7 node-insertions. The pattern of node-insertions is periodic, with period 6.

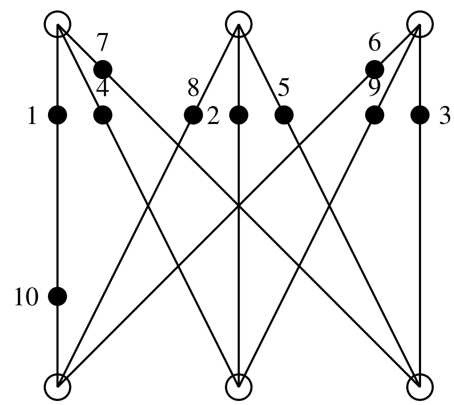

Fig. 3. Graph G_n for $n = 16$, after 10 node-insertions. The pattern of node-insertions is periodic, with period 9.

Definition 3. *$G \in \Omega(n, e)$ is* uniformly most-reliable graph *(UMRG), if $R_G(\rho) \geq R_H(\rho)$ for all $\rho \in [0, 1]$ and all $H \in \Omega(n, e)$.*

If $m_k(G) \leq m_k(H)$ for all k and all $H \in \Omega(n, e)$, then G is a UMRG. The converse is still an enignatic conjecture [22]. From Definition 3, it is clear the trees and elementary cycles are UMRG, for the respective cases $e = n - 1$ and $e = n$. The first non-trivial UMRG were provided in [24]. There, the authors show that if we insert nodes in $\theta_{2,2,2}$ as equal as possible, we obtain a sequence of UMRG for $e = n + 1$, called balanced θ-graphs. This sequence is denoted $\{\theta_n\}_{n \geq 6}$, and it is unique up to isomorphism. Further, the authors also studied the case $e = n+2$, for all $n \geq 6$. If we insert nodes in the respective edges of the complete graph K_4 picking disjoint perfect matchings in order, the sequence $\{E_n\}_{n \geq 6}$ of UMRG is obtained for $n \geq 6$. The authors conjectured that a similar node-insertion starting from the complete bipartite graph $K_{3,3}$ works to obtain the full list of UMRG when $e = n + 3$. Wang formally proved in a foundational work that the conjecture is true [25]. Wang sequence is here denoted as $\{G_n\}_{n \geq 6}$. Fig. 2 and 3 illustrate the resulting sequences of UMRG. The sequences θ_n, E_n and G_n are unique up to isomorphism.

III. Uniformly Least Reliable Graphs

Petingi et al. studied the antipodal problem: find the uniformly least reliable graphs (ULRG), which are defined analogously to UMRG [26]. The authors considered a reliability-increasing transformation independently discovered by Kelmans [27] and Satyanarayana et al. [28], known as *swing surgery*. This transformation when used *in reverse* results in a threshold graphs [29]. As a consequence, a particular sub-family of ULRG such that $e \geq (n-1)(n-2)/2+1$ is obtained. Specifically, the ULRG when $e=(n-1)(n-2)/2+r$ for some $r : 1 \leq r < n$ is the complete graph on $n-1$ nodes, K_{n-1}, and an additional node v connected to precisely r nodes belonging to K_{n-1}. This sub-family of threshold graphs is called *balloon graphs*, denoted $B_{n,e}$. The ATR evaluation of balloon graphs accepts a polynomial-time evaluation, and it serves as a universal lower bound for dense graphs [30]. Boesch et al. [20] conjectured that an extended set of graphs, here called *generalized balloon graphs*, are ULRG for all the pairs (n, e). There is some evidence that supports this conjecture. Petingi et al. [26] showed that generalized balloon graphs have the least number of spanning trees, which is a necessary condition for a graph to become ULRG. Satyanarayana et al. formally showed that this family has the least H-vector [28].

Here, additional evidence that supports the conjecture posed by [20] is given. In particular, ULRG must contain the maximum number of bridges (Proposition 1), and generalized balloon graphs consistently maximize this number (Proposition 2). Additionally, ULRG are invariant under bridge-contractions (Proposition 3). Conjecture 2 is proposed which, if affirmative, implies Boesch conjecture. A proof for particular cases is included, and justifies the interest for the stronger conjecture. Additionally, in Section IV it is formally proved that generalized balloon graphs $G_{n,e}$ are ULRG when $e \leq n+3$; see Theorem 2. Therefore, the extremal cases of sparse and dense graphs are already covered.

Denote for short $[n]=\{1,\ldots,n\}$, where n is an arbitrary positive integer, and $E=\{(a,b) : a,b \in [n], a<b\}$. Consider the following strictly-ordered relation $\prec$ in E:

$$(a,b) \prec (a',b') \leftrightarrow (b<b') \vee (b=b' \wedge a<a') \qquad (2)$$

The reader can appreciate that $\prec$ defines a strictly-ordered relation, with a single chain $\mathcal{C}=\{(1,2) \prec (1,3) \prec (2,3) \prec (1,4) \prec (2,4) \prec (3,4) \prec \ldots \prec (n-2,n) \prec (n-1,n)\}$, and the set E is precisely the edges of a complete graphs K_n. Then, the relation $\prec$ is just an order of the edges in a complete graph.

Definition 4. *Given a pair of positive integers n and e $\geq n-1$, the generalized balloon graph $G_{n,e}$ is inductively defined as the smallest set that satisfies the following clauses:*

- *If e $=n-1$, $G_{n,e}$ consists of a star-graph $K_{1,n-1}$ with node-set $[n]$ and central node $1 \in [n]$.*
- *Otherwise, pick the first element $e_c \in \mathcal{C} : e_c \notin G_{n,e}$, and $G_{n,e+1}=G_{n,e} \cup \{e_c\}$.*

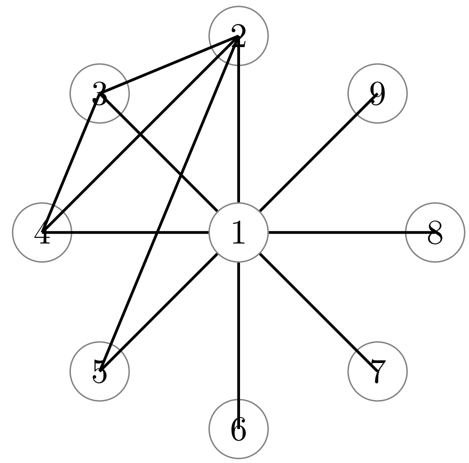

Fig. 4. Generalized Balloon Graph $G_{n,e}$ with $n=9$ nodes, $e=12$ edges and $i=12-9+1=4$ edge-insertions of the star-graph $K_{1,8}$. The edge-insertions are respectively $(2,3),(2,4),(3,4)$ and $(2,5)$.

An illustrative example is sketched in Fig. 4.

Conjecture 1 (Boesch et al. [20])**.** *Generalized balloon graphs $G_{n,e}$ are ULRG.*

Recall that the edge-contraction can produce a multigraph; however, the contraction of a bridge is always a simple graph. From Equation (1), if $m_k(G) \geq m_k(H)$ for all k and all H, then G is ULRG. It is not known if this sufficient criterion is necessary. However, this novel criterion is necessary:

Proposition 1. *If a graph $G \in \Omega(n,e)$ is ULRG, it must contain the maximum number of bridges.*

Proof. Observe that $m_1(G)=b(G)$ is the number of bridges. Consider an arbitrary $H \in \Omega(n,e)$ with $b(H)$ bridges, and assume for a moment that $b(H)>b(G)$. Since both G and H are connected, $m_0(G)=m_0(H)=0$, and by Equation (1) we get that:

$$\lim_{\rho \to 0^+} \frac{U_G(\rho)-U_H(\rho)}{\rho} = b(G)-b(H) < 0. \qquad (3)$$

In particular, this means that $U_G(\rho) < U_H(\rho)$ in some neighborhood of 0, which is a contradiction. □

Proposition 2. *The graphs $G_{n,e}$ have the maximum number of bridges $b(n,e)$ among all graphs G in the set $\Omega(n,e)$.*

Proof. By induction on the number $e \geq n-1$. The base-step holds, since $K_{1,n-1}$ is a tree, and $b(n,n-1)=n-1$. Clearly, $b(n,e) \leq n-1$ is an upper-bound, and the sequence $b_h=\{b(n,h)\}_{h \geq n-1}$ is non-increasing. Assume that the result holds for $G_{n,h}$ such that $h \geq n-1$, and consider $G_{n,h+1}=G_{n,h} \cup \{e_c\}$. There are two possibilities:

- The main block from $G_{n,h}$ is not a clique: in this case $b_{h+1}=b_h$, and by inductive hypothesis this is the maximum value, since $b_{h+1} \leq b_h$ for all h, and b_h is maximum.
- The largest block from $G_{n,h}$ is a clique. By the inductive hypothesis, this is the maximum value for b_h, and the graph is critical, in the sense that the number of bridges

is reduced under any possible addition. The best case is $b_{h+1} = b_h + 1$, which is achieved by the graph $G_{n,h+1}$.

To find $b(n,e)$, consider the co-rank $e' = e - n + 1 \geq 0$. It is clear that $b(n, n-1) = n-1$ and $b(n,n) = n-3$ (since a triangle is obtained), so we consider $e' \geq 2$. The main block of $G_{n,e}$ is periodically a clique, after a number of additions that follow an arithmetic progression. Let k be the first positive integer such that $\sum_{i=1}^{k} i = k(k+1)/2 \geq e' + 2$. The maximum number of bridges is $b(n,e) = n - k - 1$. An algebraic manipulation of the previous quadratic polynomial in k yields $b(n,e) = n - 1 - \lceil \sqrt{2(e-n+3)} - \frac{1}{2} \rceil$, when $e > n-1$, and $b(n, n-1) = n-1$ otherwise. □

Lemma 1. *A generalized balloon graph can be transformed into a balloon graph after a finite number of bridge-contractions.*

Proof. This result follows from the inductive definition of generalized balloon graphs. Observe that the block of the generalized balloon graph is precisely a balloon, except when the block is a clique (in which case it suffices to preserve one bridge and contract all the others). □

Corollary 1. *The only candidates of ULRG are generalized balloon graphs, or bridges added to a balloon graph, if exist.*

Proof. By Proposition 2, these graphs have the maximum number of bridges, which is a necessary criterion. By Lemma 1, the corresponding bridge-contractions uniquely determine a balloon graph, which is ULRG. If we consider another candidate $G \in \Omega(n,e)$, it must have $b(n,e)$ bridges. After repeated bridge-contractions, its reliability is greater than a balloon graph, since balloons are ULRG. □

Proposition 3. *If $G \in \Omega(n+1, e+1)$ has a bridge b and G is ULRG, then $G * b \in \Omega(n,e)$ is ULRG.*

Proof. We know that

$$U_G(\rho) \geq U_{G'}(\rho), \forall G' \in \Omega(n+1, e+1). \tag{4}$$

Since $G - b$ is disconnected $U_{G-b}(\rho) = 1$. Conditioning on the two possible states for b yields:

$$U_G(\rho) = \rho + (1-\rho) U_{G*b}(\rho). \tag{5}$$

Consider an arbitrary graph $H \in \Omega(n,e)$, and add a node hanging by some bridge b'. The resulting graph is H' belongs to the set $\Omega(n+1, e+1)$. Conditioning on the states for b' yields:

$$U_{H'}(\rho) = \rho + (1-\rho) U_H(\rho) \tag{6}$$

Replacing (5) and (6) into (4), and choosing $G' = H'$, we get that $U_{G*b}(\rho) \geq U_H(\rho)$, as desired. □

Conjecture 2. *If $G \in \Omega(n+1, e+1)$ has a bridge b. Then, G is ULRG if and only if $G * b$ is ULRG.*

The direct is already proved in Proposition 3. Particular cases for the converse are proved in the Appendix.

Theorem 1. *Conjecture 2 implies Boesch conjecture.*

Proof. Consider a generalized balloon graph. If it is a balloon, it is already ULRG. If not, by Lemma 1, it can be obtained by iterative additions of bridges. The result follows after repeated application of Conjecture 2, and generalized balloon graphs are ULRG. □

Brown et al. formally proved that the reliability of balloon graphs accepts a polynomial-time evaluation [30]. We have the following:

Corollary 2. *Generalized balloon graphs accept a polynomial-time reliability evaluation.*

Proof. Consider an arbitrary generalized balloon graph $G_{n,e}$, and its corresponding balloon graph $B_{n-b(n,e), e-b(n,e)}$, that is obtained after $b(n,e)$ bridge-contractions. By Proposition 2, we know this number $b(n,e)$. By Lemma 1:

$$R_{G_{n,e}}(\rho) = R_{B_{n-b(n,e),e-b(n,e)}}(\rho) \times (1-\rho)^{b(n,e)}. \tag{7}$$

□

By Corollary 1, the universal lower bound $l_b^{(n,e)}(\rho)$ for the reliability in $\Omega(n,e)$ is given by Expression (7). In the following, we study the optimality of these universal lower-bounds, showing that generalized balloon graphs are in fact ULRG whenever $e \leq n+3$.

IV. Universal Reliability Bounds

Here we cover the cases of ULRG for $e \leq n+3$. The respective UMRG are known (see Section II). Then, we find closed forms for the best universal bounds $l_b^{(n,e)}$ and $u_b^{(n,e)}$.

A. Universal Lower Bounds

Here we prove that generalized balloon graphs are ULRG whenever $e \leq n+3$. The approach is based on counting cuts. Alternatively, we will count the complement in some cases, called the k-operational subgraphs $f_k(G) = \binom{e}{k} - m_k(G)$.

Lemma 2. *Let Ω^i be the set of all simple graphs with fixed co-rank i. If $G \in \Omega^i$ minimizes $f_k(G)$ for some $k \geq 2$, it must minimize the length of all its chains as well.*

Proof. Consider a graph $G \in \Omega^i$ that minimizes $f_k(G)$ for some $k \geq 2$. Let $\mathcal{C}$ be a chain belonging to G such that $|\mathcal{C}| \geq 2$, and $e \in \mathcal{C}$. By sum-rule, the number of k-operational graphs meet the following equality:

$$f_k(G) = f_{k-1}(G-e) + f_k(G*e), \tag{8}$$

which means that the edge e is either included or not in an arbitrary k-operational set. Since $|\mathcal{C}| \geq 2$ and $k \geq 2$, we can pick a different edge $e' \in C$ such that $(G-e)-e'$ is disconnected, since all the graphs are disconnected if we remove two edges from the same chain. Therefore, $f_{k-1}(G-e) > 0$, and from Equation (8), $f_k(G)$ is strictly greater than $f_k(G*e)$. Finally, observe that G and $G*e$ have identical co-rank. If $G*e$ is a simple graph, then a contradiction is met, since by hypothesis $f_k(G)$ is minimum. Therefore, G must minimize the length of all its chains among simple graphs with identical co-rank, and the result follows. □

A subtlety related to Lemma 2 is in order. Consider a θ-graph $\theta_{2,2,1}$. It has two endpoints joined by three chains whose

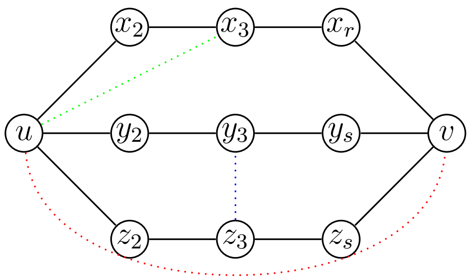

Fig. 5. Three distinguished cases: (I) connect endpoints u and v, (II) connect an endpoint u and an internal node (x_3 in the example) and (III) connect two internal nodes (y_3 and z_3 in the example).

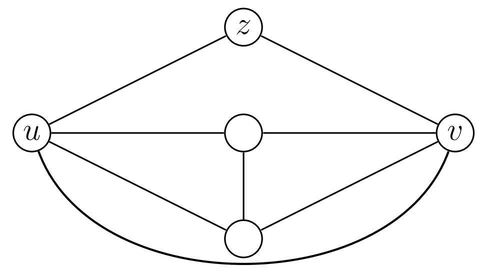

Fig. 6. Addition of two open-ears to some θ-graph with extremes u and v, to obtain a simple graph with minimum values for f_2 and f_3. All the chains are trivial (i.e., single edges), except for the chain $\mathcal{C} = \{(u,z),(z,v)\}$, that is required to avoid parallel edges.

lengths are respectively 2, 2 and 1 (see Figure 1 for a sketch of θ-graphs). Apparently, in the proof of Lemma 2 we can contract an edge of some chain $\mathcal{C}$ whenever $|\mathcal{C}| \geq 2$. However, we cannot contract edges belonging to $\theta_{2,2,1}$, since it leads to a multigraph, and Ω^i contains simple graphs only.

A celebrated work pioneered by Whitney states that every biconnected graph can be constructed by iterative additions of open ears, starting from an elementary cycle [31]. The following technical lemmas will be useful:

Lemma 3. *If $G \in \Omega(n, n+2)$ is biconnected, then $f_2(G) \geq 15$, with equality if and only if $G = K_4$.*

Proof. By [31], we know that G consists of the addition of two open-ears to some elementary cycle. After the addition of a single open-ear to an elementary cycle, a θ-graph is always obtained. Therefore, G consists of an open-ear added to θ_{l_1,l_2,l_3}, for some lengths l_1, l_2 and l_3. The open-ear can either connect the two endpoints, one of them, or none. The resulting graphs have correspondingly four, five or six chains, and by Lemma 2, we must choose the shortest lengths in order to minimize f_2. Let us discuss the three cases separately (see Figure 5 for an illustration of the different cases):

- If the open-ear connects both endpoints, a generalized θ-graph θ_{l_1,l_2,l_3,l_4}, is obtained. The shortest lengths that determine a simple graph are $l_1 = l_2 = l_3 = 2$, and $l_4 = 1$ (since we deal with simple graphs only). Then, $f_2(G) \geq f_2(\theta_{2,2,2,1}) = 3 \times (2 \times 2 + 2 \times 1) = 18$.
- If the open-ear connects only one degree-3 node, five chains are obtained (two configured in parallel). The shortest chains that determine a simple graph have lengths $2, 2, 1, 1, 1$, and $f_2(G) \geq 2 \times 2 + 6 \times 2 \times 1 + 3 \times 1 = 19$.
- If the open-ear connects internal nodes, the resulting graph has six chains. The smallest value for f_2 is achieved choosing all trivial chains, and the result is K_4, so $f_2(G) \geq \binom{6}{2} = 15$.

It can be concluded that $f_2(G) \geq 15$, and the minimum is obtained if and only if $G = K_4$. □

Lemma 4. *If $G \in \Omega(n, n+3)$ is biconnected, then $f_2(G) \geq 27$ and $f_3(G) = 48$, with equalities if and only if G consists of K_5 minus two adjacent edges.*

Proof. Combining the chain minimalities from Lemma 2 and an analogous discussion with two ears added to some θ-graph, the minimum values for f_2 and f_3 are found when one open-ear connects the endpoints and the other open-ear connects internal nodes; see Fig. 6. Observe that some chains have length equal to two, since the resulting graph must be simple. The result is precisely K_5 minus two adjacent edges. □

Theorem 2. *Generalized balloon graphs are ULRG whenever $e \leq n+3$.*

Proof. We will prove a stronger result: $m_k(G_{n,e}) \geq m_k(G)$, for all $k \in \{0, \ldots, e\}$ and all $G \in \Omega(n, e)$, if $e \leq n+3$. We denote for convenience $\Omega^i = \Omega(n, n-1+i)$. Let us cover the cases $i \in \{0, \ldots, 4\}$. Observe in general that the k-cuts from the members belonging to Ω^i meet the following relations:

- $m_0 = 0$ in all the cases (since the graphs are connected).
- $m_1 = b(n, n-1+i)$, the number of bridges, is maximized in the generalized balloon graphs.
- $m_i = \binom{n-1+i}{i} - \tau$, being τ the tree-number. Since the tree-number is minimized in the generalized balloon graphs, m_i is maximized in these graphs.
- $m_j = \binom{e}{j}$ for $j > i$, in all the graphs (since connected graphs must have at least $n-1$ edges).

By the previous relations, we can see that $G_{n,e}$ maximizes all the k-cuts when $i \in \{0,1,2\}$. In particular, these graphs are ULRG in Ω^i for $i \in \{0,1,2\}$. It suffices to prove that $G_{n,e}$ is max-m_k for $i \in \{3,4\}$. Let us study separately both cases.

If $i = 3$, from the previous relations, it suffices to see that $f_2(G_{n,e}) \leq f_2(G)$ for all $G \in \Omega^3$. Since f_2 is invariant under bridge-contractions, we can compare their non-trivial blocks: K_4 versus $G' \in \Omega(n, n+2)$, obtained from G after bridge-contractions. The result follows from Lemma 3.

Finally, if $i = 4$ we need to study f_2 and f_3. The main block from $G_{n,e}$ is K_5 minus two adjacent edges. A straight counting leads to determine that $f_2(G_{n,e}) = 27$ and $f_3(G_{n,e}) = 48$. Consider an arbitrary $G \in \Omega^4$. If we contract all its bridges, we get a bridgeless graph $G' \in \Omega(n, n+3)$. If G' has several blocks, there are two cases:

1) G' includes some block $G_1 \in \Omega(n, n+3)$. In this case, the remaining blocks must be elementary cycles, and the coefficients f_2 and f_3 are strictly greater than an individual block, in which $G = G_1$.

2) G' includes a couple of blocks $G_1 \in \Omega(n, n+1)$ and $G_2 \in \Omega(n, n+2)$. Based on Lemma 3, the graphs with smallest coefficients f_i for both cases are $G_1 = \theta_{2,2,1}$ and $G_2 = K_4$. The coefficients f_2 and f_3 from G are large compared with $G_{n,e}$, since:

$$\begin{aligned} f_2(G) &\geq f_2(G_1 \cup G_2) \\ &= f_1(K_4)f_1(\theta_{2,2,1}) + f_2(K_4) + f_2(\theta_{2,2,1}) \\ &= 6 \times 5 + 15 + 8 = 53 > 27 = f_2(G_{n,e}); \\ f_3(G) &\geq f_3(G_1 \cup G_2) \\ &= f_1(K_4)f_2(\theta_{2,2,1}) + f_2(K_4)f_1(\theta_{2,2,1}) + f_3(K_4) \\ &= 6 \times 8 + 15 \times 5 + 16 = 139 > 48 = f_3(G_{n,e}). \end{aligned}$$

Finally, if G' consists of a single block, G' is biconnected, and the result follows from Lemma 4. □

Supported by Theorem 2 and Expression (7), we get the best universal lower-bounds $l_b^i(\rho)$ for each $\Omega^i = \Omega(n, n-1+i)$:

$$l_b^0(\rho) = (1-\rho)^{n-1}; \tag{9}$$

$$l_b^1(\rho) = (1-\rho)^{n-3}((1-\rho)^3 + 3\rho(1-\rho)^2); \tag{10}$$

$$l_b^2(\rho) = (1-\rho)^{n-4}((1-\rho)^5 + 5\rho(1-\rho)^4 + 8\rho^2(1-\rho)^3); \tag{11}$$

$$\begin{aligned} l_b^3(\rho) = (1-\rho)^{n-4}&((1-\rho)^6 + 6\rho(1-\rho)^5 \\ &+ 15\rho^2(1-\rho)^4 + 16\rho^3(1-\rho)^3); \end{aligned} \tag{12}$$

$$\begin{aligned} l_b^4(\rho) = (1-\rho)^{n-5}&((1-\rho)^8 + 8\rho(1-\rho)^7 \\ &+ 27\rho^2(1-\rho)^6 + 48\rho^3(1-\rho)^5) + 40\rho^4(1-\rho)^4 \end{aligned} \tag{13}$$

Observe that l_b^0 and l_b^1 stand for the reliabilities of a tree and a triangle with bridges, while l_b^i are the reliabilities of bridges together with $K_4 - \{e\}$, K_4 and K_5 minus two adjacent edges, for the respective cases $i \in \{2, 3, 4\}$.

B. Universal Upper Bounds

Trees and elementary cycles are UMRG in the sets Ω^0 and Ω^1, with a straight reliability calculation:

$$u_b^0(\rho) = (1-\rho)^{n-1}; \tag{14}$$

$$u_b^1(\rho) = (1-\rho)^{n-1}((1-\rho) + n\rho). \tag{15}$$

The graph-sequences θ_n, E_n and G_n are respectively UMRG in the sets Ω^i for $i \in \{2, 3, 4\}$. The three sequences are obtained by an iterative node-insertion process, in different edges. First, note that all the graphs are biconnected, and $m_0 = m_1 = 0$.

In order to count m_k for $k \geq 2$, the key is to observe that k-cuts are obtained when we pick at least two edges from the same chain, or k edges from different chains that disconnect the original graph with if subdivisions were applied (respectively, $\theta_{2,2,2}$, K_4 or $K_{3,3}$). A unified expression for m_k will be here introduced. First, consider the following terminology:

- Denote $O_{n,i}$ the optimal (UMRG) sequence in the set $\Omega^i = \Omega(n, n-1+i)$. In particular, $O_{n,2} = \theta_n$, $O_{n,3} = E_n$ and $O_{n,4} = G_n$.
- The distillation of $O_{n,i}$ is denoted by $D(O_{n,i})$. Clearly, $D(O_{n,2}) = \theta_{1,1,1}$, $D(O_{n,3}) = K_4$ and $D(O_{n,4}) = K_{3,3}$.
- Denote r_i the number of chains in the graphs $O_{n,i}$: $r_2 = 3$, $r_3 = 6$ and $r_4 = 9$.
- Order the chains $l_1, \ldots, l_{r_i}$ according to the insertion-order in the first round of the corresponding UMRG (see Subsection II-B for details).
- Define the naturals $k_1, \ldots, k_{r_i}$ that represent the number of edges to be removed from the corresponding chain $l_1, \ldots, l_{r_i}$.
- If $\vec{k} = (k_1, \ldots, k_{r_i})$ is a binary word ($\vec{k} \in \{0,1\}^{r_i}$), the graph $D(O_{n,i})^{\vec{k}}$ is the distillation $D(O_{n,i})$, after the remotion of the simple edges l_i (the trivial chains) such that $k_i = 1$.
- The indicator function 1_x equals 1 if and only if x is true; 0 otherwise.

Observe that if we pick at least two edges from the same chain, the resulting graph is disconnected. Alternatively, if we pick either zero or one edge from each chain according to the vector $\vec{k}$, the resulting graph is disconnected if and only if the distillation $D(O_{n,i})^{\vec{k}}$ is disconnected. As a consequence:

$$\begin{aligned} m_k(O_{n,i}) = &\sum_{k_1+\ldots+k_{r_i}=k:\exists j:k_j\geq 2} \prod_{i=1}^{r_i} \binom{l_i}{k_i} \\ &+ \sum_{\vec{k}\in\{0,1\}^{r_i}:k_1+\ldots+k_{r_i}=k} \prod_{i:k_i=1} l_i 1_{\{D(O_{n,i})^{\vec{k}} disconnected\}}. \end{aligned} \tag{16}$$

Expression (16) provides a unified framework to count the k-cuts for all the sequences under study, after the replacement into the unreliability from Equation (1):

$$\begin{aligned} U_{O_{n,i}}(\rho) &= \sum_{k=2}^{n-1+i} m_k(O_{n,i})\rho^k(1-\rho)^{n-1+i-k} \\ &= \sum_{k=2}^{i} m_k(O_{n,i})\rho^k(1-\rho)^{n-1+i-k} \\ &+ \sum_{k=i+1}^{n-1+i} \binom{n-1+i}{k}, \end{aligned} \tag{17}$$

where the last equality uses the fact that graphs are disconnected if they have less than $n-1$ edges. Finally, $R_{O_{n,i}}(\rho) = 1 - U_{O_{n,i}}(\rho)$, and since $O_{n,i}$ are UMRG, we get universal upper-bounds for $i \in \{2, 3, 4\}$ as well:

$$u_b^i = 1 - U_{O_{n,i}}(\rho), \forall i \in \{2, 3, 4\}. \tag{18}$$

Since $r_i \in \{3, 6, 9\}$ respectively for $i \in \{2, 3, 4\}$ and the maximum non-trivial m_k is met when $k = k_{max} = 4$, the summations from Expression (16) can be computed efficiently for every member of each infinite sequence of the considered UMRG, no matter how large is the number of nodes n. In fact, the maximum number of terms involved as a whole in both summations occur when $r_i = 9$ and

$k = 4$. This is precisely the number of elements in the set $\{(k_1, \ldots, k_9) \in \mathbb{N}^9 : \sum_{i=1}^{9} k_i = 4\}$, which is $\binom{12}{4} = 495$.

As an example, for θ_n it suffices to count $m_2(\theta_n)$ and insert the value in Expression (17); note that the other coefficients m_k are all known beforehand. If we remove two edges from different chains, the subgraph remains connected, and the last summation from (16) is null. If the lengths of the three chains are l_1, l_2 and l_3, we get that

$$m_2(\theta_{l_1,l_2,l_3}) = \binom{l_1}{2} + \binom{l_2}{2} + \binom{l_3}{2}. \quad (19)$$

The number $m_2(\theta_n)$ is the particular evaluation of $m_2(\theta_{l_1,l_2,l_3})$, where $l_1 \geq l_2 \geq l_3$ are selected in such a way that $l_1 + l_2 + l_3 = n + 1$. Similar replacements for E_n or G_n provide the best universal upper-bounds for the ATR.

The best universal reliability-bounds $[l_b^i(\rho), u_b^i(\rho)]$ are obtained for the respective sets $\Omega^i = \Omega(n, n-1+i)$, and any $i \in \{0, \ldots, 4\}$, combining Expressions (9)-(13) with (14)-(15) and (18).

V. Cost-Reliability Trade-off

In practice, the cost is related to the number of edges (or the distance in a physical deployment of FTTH for example). Naturally, if we are given n nodes, the least number of edges to meet connectivity is $e = n - 1$, and resulting networks are trees. If we consider $e = n - 1 + i$ edges instead for some positive integer i, the *marginal cost* is precisely the number of additional edges i.

A connectivity-driven approach is to consider the *benefit* as $\lambda(G)$, the network connectivity, given its importance in communication systems. The *utility function* is then $u(G) = \lambda(G) - i$, being i is the marginal cost using $e = n - 1 + i$ edges (or the co-rank of a graph). A celebrated work authored by Harary constructs graphs with the maximum connectivity, given a prescribed number of nodes and edges [32]. Therefore, Harary graphs have the largest connectivity; however, they have several edges (i.e., cost). Curiously enough, the only graphs with the greatest utility are precisely trees and elementary cycles [33]. This connectivity-driven approach is pessimistic, and it confirms that the only way to increase both the number of edges and connectivity is when an edge connects the endpoint of a path, obtaining an elementary cycle (this is the exceptional case where the connectivity is increased using a single edge).

Our approach is reliability-driven. We want to analyze two effects:

- Determine how relevant is to choose a smart topology. This is considered in terms of the gap between ULRG and UMRG.
- Understand the sensibility of the reliability with respect to edge-additions.

The following merit functions will be considered for our purposes.

Definition 5. *The* gap-function *in* $\Omega^i = \Omega(n, n-1+i)$ *is* $\delta^i(\rho) = u_b^i(\rho) - l_b^i(\rho)$.

The best universal bounds u_b^i and l_b^i were derived in Section IV, for each $i \in \{0, \ldots, 4\}$.

Definition 6. *The* maximum gap *is the infinite-norm of the gap-function:* $\delta^(i)_{max} = max_{x \in [0,1]}\{\delta^i(\rho)\} = \|\delta^i\|_\infty^{[0,1]}$, *and the critical point is the probability* ρ_c^i *such that* $\delta^i(\rho_c^i) = \delta_{max}^i$.

The critical point is the failure probability where the gap between the ULRG and UMRG is maximum. If a practical network is working on the critical point, the network designer should adequately select the links for the underlying infrastructure of the system, since the discrepancy between the worst and the best network is considerable (i.e., the maximum gap).

Let us find analytically the critical point and maximum gap for particular cases. The case where $i = 0$ is straight, since trees are both ULRG and UMRG, and the gap is null. The gap-function $\delta^1(\rho)$ is the difference between Equations (10) and (14):

$$\delta^1(\rho) = (n-3)\rho(1-\rho)^{n-1}, \ \forall n \geq 3. \quad (20)$$

Solving $(\delta^1)'(\rho) = 0$ we can find the critical point $\rho_c^1 = \frac{1}{n}$, for all $n \geq 3$. Therefore, *the critical point occurs under highly-reliable components*. This interesting result highlights that highly-reliably systems have a critical design, specially where the failure probability of the individual components represents a rare event in networks with a massive number of terminals. Let us find the maximum gap:

$$\delta_{max}^i(n) = \delta^1(\rho_c^1) = \frac{n-3}{n}\left(1 - \frac{1}{n}\right)^{n-1} \quad (21)$$

The maximum gap is non-vanishing with the network size. In effect:

$$\lim_{n\to\infty} \delta_{max}^i(n) = \lim_{n\to\infty} \frac{n-3}{n-1}\left(1 - \frac{1}{n}\right)^{n} = \frac{1}{e} > 0; \quad (22)$$

this means that an incorrect network design is even worse when the number of terminals is increased. Similarly, the gap-function when $i = 2$ is:

$$\delta^2(\rho) = (1-\rho)^{n-1}[(1-\rho)\rho(n-4) + \rho^2(f_2(\theta_n) - 8)], \ \forall n \geq 5. \quad (23)$$

The critical point is obtained analogously, solving the quadratic equation $(\delta^2)'(\rho) = 0$. If we consider the asymptotic behaviour where $f_2(\theta_n) \sim \frac{n^2}{3}$, the maximum gap is monotonically increasing with the number of terminals, with limit $(1 + 2k/3)e^{-k}$ being $k = (\sqrt{13} - 1)/2$, and the critical point is asymptotically inverse with the number of terminals: $\rho_c^2 = k/n$.

A numerical analysis confirms that this pattern is preserved for $i \in \{3, 4\}$. Fig. 7-8 illustrate the gap functions $\delta^i(\rho)$ for the respective cases $i \in \{3, 4\}$ and particular values of terminals n. The reader can appreciate that the gap-function assumes large values when the edge-failure probability ρ is close to zero. These results suggest that an incorrect network design has a catastrophic effect, specially under highly-reliable systems, where a large difference between ULRG and UMRG is found.

Another attractive element for the network designer is to determine the sensibility in the system reliability under edge additions:

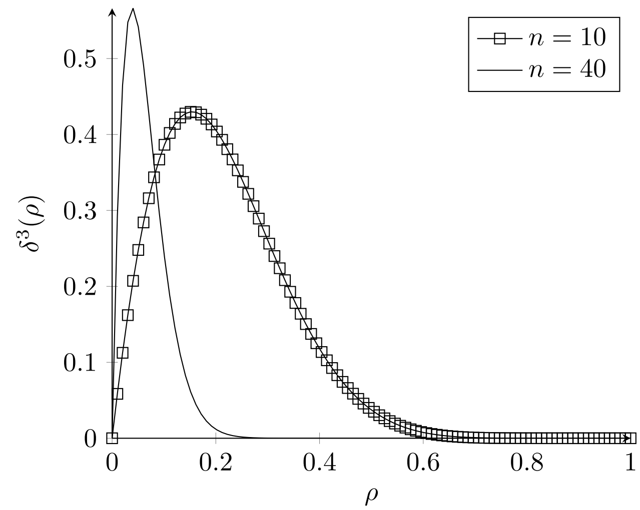

Fig. 7. Gap Function $\delta^3(\rho)$. The mass of the function concentrates in $\rho = 0$ when the number of terminals n is increased.

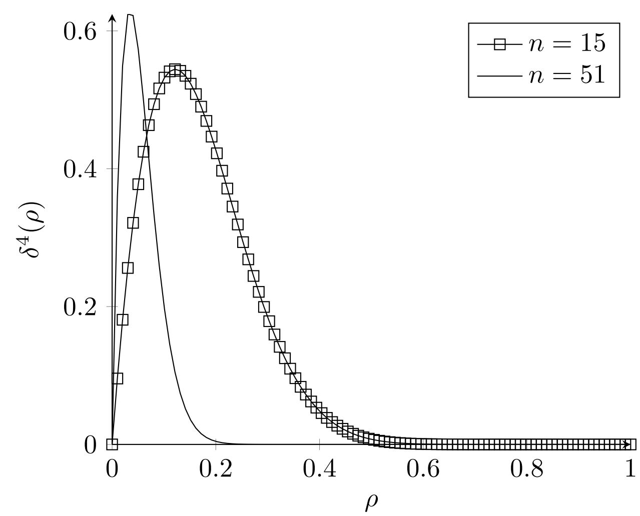

Fig. 8. Gap Function $\delta^4(\rho)$. The mass of the function concentrates in $\rho = 0$ when the number of terminals n is increased.

Definition 7. *The* sensibility function *is:*

$$s^{i,i+1}(\rho) = u_b^{i+1}(\rho) - u_b^i(\rho).$$

Observe that the sensibility function is not associated with network augmentation, since UMRG are not produced using iterative additions [34]. However, it makes sense if we can choose the number of links for any fixed number of terminals.

The following parameters are associated to edge-additions:

Definition 8. *The* maximum sensibility *is the number* $s_{max}^{i,i+1} = max_{x\in[0,1]}\{s^{i,i+1}(\rho)\}$*, and the best point for an edge-addition is the probability* $\rho_s^{i,i+1}$ *such that* $s^{i,i+1}(\rho_s^{i,i+1}) = s_{max}^{i,i+1}$.

Let us determine the sensibility function and its main parameters for particular cases:

$$\begin{aligned} s^{0,1}(\rho) &= R_{C_n}(\rho) - R_{P_n}(\rho) \\ &= [(1-\rho)^n + n\rho(1-\rho)^{n-1}] - [(1-\rho)^{n-1}] \\ &= (n-1)\rho(1-\rho)^{n-1}. \end{aligned}$$

It is straight to see that the best point for an edge-addition is $\rho_s^{0,1} = \frac{1}{n}$. Interestingly, the best point for an edge-addition is precisely the critical point. If we substitute using $\rho = 1/n$, the maximum sensibility is obtained:

$$s^{0,1}(1/n) = \left(1 - \frac{1}{n}\right)^n,$$

which converges again to $1/e$. An analogous reasoning lead us to determine

$$s^{1,2}(\rho) = R_{\theta_n}(\rho) - R_{C_n}(\rho) = \rho^2(1-\rho)^{n-1}(f_2(\theta_n) - n).$$

Immediately we get $\rho_s^{0,1} = \frac{2}{n+1}$, and the maximum sensibility converges to $4/3e^{-2}$.

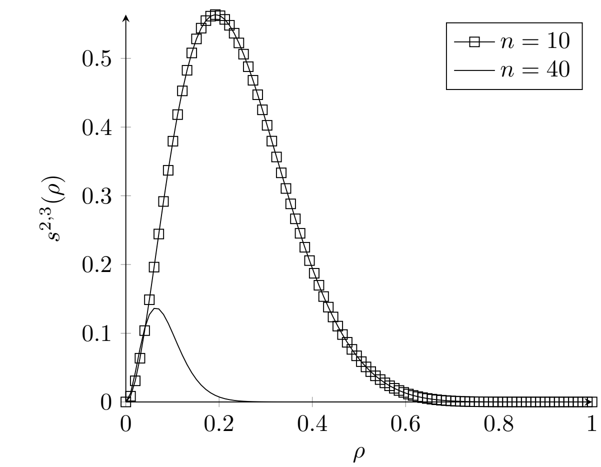

Fig. 9. Sensibility Function $s^{2,3}(\rho)$. The mass of the function concentrates in $\rho = 0$ when the number of terminals n is increased.

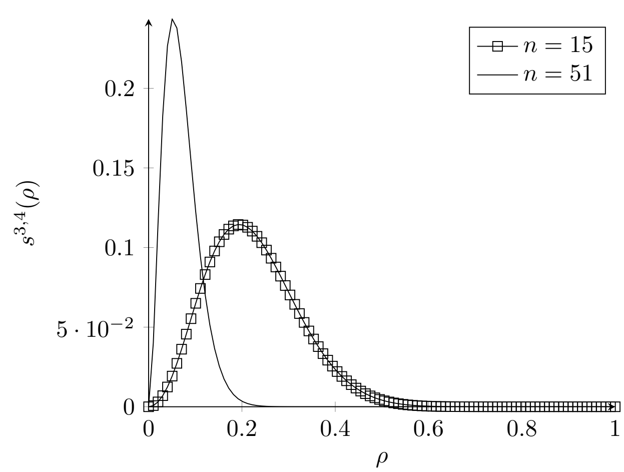

Fig. 10. Sensibility Function $s^{3,4}(\rho)$. The mass of the function concentrates in $\rho = 0$ when the number of terminals n is increased.

A *concentration-phenomenon* towards the rare-event is notorious, in terms of criticality and sensibility as well.

This pattern is also verified numerically for $i \in \{3, 4\}$, as Fig. 9-10 respectively show. As a synthesis, it can be concluded that a smart network design is essential for highly-reliable systems, where the elementary link failures represent a rare event. This situation is specially relevant when the number of terminals under communication is large enough.

VI. Conclusion

The hardness of the all-terminal reliability (ATR) evaluation promotes the development of estimation techniques and reliability bounds. The literature offers several bounds for any fixed graph, mostly based on dropping terms from exact methods. As far as the author knows, this is the first work where the best universal reliability bounds are given for sparse graphs with n nodes and $e \leq n+3$ edges. To achieve this goal, it is formally proved that generalized balloon graphs are uniformly least-reliable graphs (ULRG) when $e \leq n+3$; the antipodal uniformly most reliable graphs (UMRG) are already available in the literature for those specific cases. Furthermore, Gross and Saccoman conjectured that these universal upper-bounds hold even under the extended set of multigraphs [35]. This conjecture is true, and a formal proof is recently provided in an article to appear [36].

The ATR is intrinsically related to counting, at least, when the edge-failures are independent and identical. Therefore, a cut-based representation is offered to speed-up the counting, and find efficiently the ATR.

It is worth to remark that this cut-based representation has limitations. It is suitable for infinite sequences of homeomorphic graphs, or for graphs with bounded co-rank. The reader can appreciate that all the graph-sequences here considered are homeomorphic to the first graph from the sequence. This is a key element for the success of the current technique.

Finally, the cost-reliability trade-off is here studied in terms of edge-additions. Highly-reliable systems show to be the most critical in terms of network design, where the gap between ULRG and UMRG is maximum. Further studies should be performed for dense, or mesh-networks.

Several problems deserve future work. If Conjecture 2 is correct, Boesch conjecture is true, and generalized balloon graphs are ULRG. Only partial cases were covered here. In particular, generalized balloon graphs have the maximum number of bridges $b(n, e)$ among all the connected simple graphs with n nodes and e edges; this is a necessary condition for a graph to be ULRG (a closed-form for $b(n, e)$ is here introduced). Remarkably, the set of ULRG is also closed under bridge-contractions. The literature offers ULRG and UMRG for the pairs (n, e) with extremely low (sparse) or high (almost-complete) densities. The theory of UMRG is extremely useful for network synthesis, but it is still not mature yet. An outstanding progress in the construction of reliability-increasing transformations was the introduction of Swing Surgery, and its applicability in reverse [37]. Finding new reliability-increasing transformations is also challenging.

Appendix
Converse of Conjecture 2

If Conjecture 2 is true, the study of ULRG is fully covered, and generalized balloon graphs are ULRG, as well as balloon graphs with additional bridges arbitrarily connected to the main block.

The direct of Conjecture 2 is true; see Proposition 3. Two special cases for the converse are here proved. Denote $G^b \in \Omega(n+1, e+1)$ the graph obtained by the addition of a hanging node, and its corresponding bridge b, to $G \in \Omega(n, e)$. We can rephrase the converse of Conjecture 2 as follows:

Conjecture 3. *If G is ULRG, then G^b is ULRG.*

Consider $G \in \Omega(n, e)$ such that G is ULRG. The reader can appreciate that the following statement is a particular case:

Proposition 4. *If $H \in \Omega(n+1, e+1)$ has some bridge b_h, G^b is uniformly least-reliable than H.*

Proof.

$$\begin{aligned} U_{G^b}(\rho) &= \rho + (1-\rho)U_{(G^b)*b}(\rho) \\ &= \rho + (1-\rho)U_G(\rho) \\ &\geq \rho + (1-\rho)U_{H*b_h}(\rho) = U_H(\rho); \end{aligned}$$

where the inequality uses the fact that $H * b_n \in \Omega(n, e)$ and G is ULRG. □

Proposition 4 can be strengthened:

Proposition 5. *If $H \in \Omega(n+1, e+1)$ has some edge w such that $H * w$ is simple, G^b is uniformly least-reliable than H.*

Proof. The same reasoning from Proposition 4 holds, since $H * w$ is simple and hence can be compared with G. □

The reader can appreciate that the remaining case where $H * w$ is a multigraph for all the possible edges $w \in E(H)$ is not covered yet. It is interesting to explore the applicability of swing surgery in reverse or similar transformations, in order to *weaken* H in such cases, either finding a least-reliable graph H' with a bridge, or with some edge w such that $H' * w$ is a simple graph. This is a challenging research topic for future work.

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