ORIGINAL ARTICLE

Networks

Uniformly Optimally Reliable Graphs: A Survey

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Which is the most reliable graph with n nodes and m edges? This celebrated problem has several aspects, according to the notion of optimality (in a local or uniform sense), failure type (either nodes or edges), or reliability model (all-terminal connectedness, two-terminal or multiterminal setting). This article presents a chronological survey of the multiple proposals to address the problem, together with recent trends and enigmatic conjectures posed decades ago that promote further research.

KEYWORDS
Uniformly most reliable graph, Uniformly least reliable graph, All-terminal reliability, Two-terminal reliability, Failure type, Graph theory

1 | HISTORIC MOTIVATION

Claude Shannon developed a mathematical theory of communications, which has tremendous impact in modern telecommunication systems [69]. He was also interested in optimal network design. In fact, Moore and Shannon wanted to design a *perfect* system using imperfect components [48]. This goal seems impossible; however, they formally proved that special configurations of imperfect components (i.e., relays in a circuit) converge, using infinitely many components, to a perfect system. A key concept was that of *network composition* or *self-similarity*, where the elementary components are iteratively replaced by the whole system, such as a *fractal*. They also used the deletion-contraction formula, which is an essential concept in the study of network reliability. This concept is now mature, and has been extensively studied in the literature [49, 66, 68]. Moskowitz formally proved the *factoring theorem*, which serves to repeatedly reduce/simplify the system into smaller subsystems, which can then be analyzed separately. This is a kind of *divide and conquer* notion widely used in network reliability. He also understood the importance of redundancy, and showed its strength in some series-parallel networks [49]. Satyanarayana extensively studied this factorization, showing elegant linear-time evaluation algorithms for series-parallel networks considering an invariant called the *domination of a graph*, which is defined in terms of minimally operational subgraphs [66, 68].

On the one hand, the construction of perfect systems considers *infinite* imperfect components, but in practice, the budget is always a constraint, and real systems are finite. On the other hand, a mature analysis of series-parallel networks is available in the scientific literature, but this subfamily of networks does not include the optimal design.

The mathematical evaluation of the reliability function for a given system has major importance. The literature is abundant; see [57] for an exposition on exact methods, or [64] for a book that covers approximate approaches. However, less attention has been dedicated to the practical problem of *network synthesis*: which is the most reliable network that interconnects a fixed number of nodes, using a fixed number of edges? The answer has an active impact on decision-making when we must interconnect fixed sites with a fixed budget (limited number of links).

In the mathematical models, it is common to assume that the components fail with identical and independent probability $\rho \in [0, 1]$. Unless otherwise specified, it is reasonable to consider homogeneous components with identical failures. The hypothesis of independence is dropped in some works; see [25] for a rich discussion with abundant references, and [7, 47] for a reliability optimization framework under dependent failures. Nevertheless, the identical/independent setting is a valuable mathematical abstraction, and the question will be studied in this article under these assumptions.

Do we know the precise value for the failure probability ρ beforehand? Normally not. In fact, a network is optimal in a *uniform sense* if its reliability is greater than or equal to that of all other networks with identical numbers of nodes and edges, regardless of the specific value of ρ (this is, *for all* the possible values of $\rho \in [0, 1]$).

There are at least three reasons to study uniformly optimally-reliable graphs. The first is a static network design: the particular value of the elementary reliabilities can be neglected, and the network is optimal in a uniform sense. The second is time-invariance: if all the components of the system uniformly deteriorate with time, the network still represents the best design. The third is that these models represent a mathematical abstraction of several real-life systems.

The search for symmetry and beauty is intriguing, and the existence, uniqueness and construction of uniformly optimally-reliable graphs frequently appear to be the main mathematical questions. We celebrate the 50th anniversary of the *Networks* journal, which has a deep relation with the progress in this field. This is not the first survey on this topic. Frank Boesch formally defined the concept of uniformly optimally-reliable graphs in a survey [15]. Nevertheless, that survey is focused on locally optimal analysis for values of ρ either close to 0 or 1. Wendy Myrvold [51] offered a brief but nice report on the progress until 1996. Frank Boesch *et al.* [14] offered another survey in 2009 that involves reliability in a wider sense, and a single section is focused on this topic. A graceful comprehensive treatment on network reliability was recently authored by Jason Brown *et al.* [25].

This survey is organized as follows. The models under study are presented in Section 2. A historic-driven analysis of the uniformly most reliable graphs (UMRGs) under the all-terminal reliability setting with edge failures is given in Section 3. The progress in network synthesis under node failures is outlined in Section 4. Section 5 covers other specific network reliability models, such as two-terminal reliability, and extends the analysis to multigraphs and uniformly least reliable graphs to find universal reliability bounds. A summary of open problems and trends for future work is given in Section 6.

2 | MODELS

In all the models under study, we are given a ground graph $G = (V, E)$ with $n = |V|$ nodes and $e = |E|$ edges, which is undirected, loopless and connected. We deal with simple graphs (i.e., no repeated edges are allowed), unless otherwise specified. Some components are subject to random independent failures. Two major cases are considered:

- Edge reliability, where nodes are perfect, but the edges fail with independent identical probability.
- Node reliability, which is analogous but considers perfect edges and imperfect nodes instead.

Let us denote $\rho \in [0, 1]$ the elementary failure probability in either case. Joint failures are not discussed here; the interested reader can consult [56].

The *reliability* $R_G(\rho)$ stands for the probability of a specified successful event, such as all-to-all connectedness (also known as the all-terminal reliability), two-terminal or more generally, multiterminal communication. It is sometimes convenient to deal with its complementary probability, called *unreliability*, or $U_G(\rho)$. Consider, for instance, the all-terminal reliability under edge failures. Let $m_k(G)$ denote the number of nonconnected subgraphs $H = (V, F)$ such that $F \subseteq E$ and $|E| - |F| = k$. The *cut-vector* is $m(G) = (m_0(G), \ldots, m_e(G))$. Note that $m_k(G)$ is the number of ways to disconnect G by precisely removing k edges. By the sum-rule, we can find $U_G(\rho)$ using the cut-vector:

$$U_G(\rho) = \sum_{k=0}^{e} m_k(G)\rho^k(1-\rho)^{e-k}. \tag{1}$$

Similar expressions are obtained under reliability models with node failures. A discussion of reliability polynomials and their roots in the complex plane in a more abstract setting of simplicial complexes and matroids is given in [24]. An excellent monograph on the combinatorics of network reliability was authored by Colbourn [31].

Observe that the unreliability polynomial $U_G(\rho)$ is fully determined by the cut-vector. Nevertheless, we warn the reader that finding the cut-vector is an intrinsically difficult task for general graphs. In fact, Valiant introduced a hierarchy of $\#P$-complete counting problems [75], basically harder problems than $\mathcal{NP}$-complete decision problems [32, 41]. It is formally proven that finding the cut-vector belongs to the class of $\#P$-complete counting problems under both edge reliability [76] and node reliability [73]. Network reliability evaluation is difficult for node reliability [73], multi-terminal [63] and two-terminal reliability models [6]. A gentle review of the complexity of network reliability analysis is given by Ball [5]. The classical book authored by Garey and Johnson provides a rich introduction to complexity theory, including valuable examples of $\mathcal{NP}$-complete decision problems on network reliability, graph theory and combinatorial optimization, among others [34]. The reader is invited to consult the book authored by Harary for graph-theoretic terminology; it also covers some open problems in graph theory and fundamental applications in engineering [39].

3 | UNIFORMLY MOST RELIABLE GRAPHS

In this section we consider the all-terminal reliability model under edge failures. In a seminal paper, Boesch introduced the corresponding synthesis problem: given n and e, find (n, e)-graphs G such that $U_G(\rho) \leq U_H(\rho)$ for all $\rho \in [0, 1]$, and all (n, e)-graphs H. Such graphs were called *uniformly optimally reliable graphs* by Boesch [15]. Myrvold later renamed this graph class the *uniformly most reliable graphs* to avoid a tongue twister [51]. We will use the latter and the corresponding acronym UMRGs in this context and other models.

From Equation (1), we can appreciate that if $m_k(G) \leq m_k(H)$ for all $k \in \{0, 1, \ldots, e\}$ and all (n, e)-graphs H, then the unreliability polynomial is uniformly dominated, that is, $U_G(\rho) \leq U_H(\rho)$ for all $\rho \in [0, 1]$, and G is UMRG. The converse is one of the most enigmatic conjectures in the field, posed in 1986 by Boesch [15].

Observe that the local optimality in a neighborhood of $\rho = 0$ and $\rho = 1$ are necessary conditions for a graph to become UMRG. On the one hand, when ρ tends to 0 the unreliability polynomial is equivalent to $m_\lambda \rho^\lambda (1 - \rho)^{e-\lambda}$, where λ is the *edge connectivity*, or the first positive integer k such that $m_k(G) > 0$. Clearly, the best choice is to pick graphs with the greatest edge connectivity λ and the least number of edge-disconnecting sets m_λ, which are called *λ-optimal graphs*. The class of λ-optimal graphs presents the best reliability in some neighborhood of 0, this is, for all $\rho \in [0, \rho_0)$. Additionally, λ-optimal graphs share similar reliability polynomials in that interval with controlled gaps; see [9] for details. On the other hand, if ρ tends to 1, the polynomial is equivalent to $m_{e-n+1}(G)\rho^{e-n+1}(1 - \rho)^{n-1}$, and it is necessary to minimize the number m_{e-n+1}. Its complement $t(G) = \binom{n}{n-e+1} - m_{e-n+1}$ is precisely the number of spanning trees, or the *tree number* of the graph G. An (n, e)-graph with the greatest tree number in its class is called a *t-optimal graph*. It is clear from its definition that UMRGs are t-optimal [77].

Even though the concepts of t-optimality and λ-optimality are more primitive than UMRGs, they are still not well understood. Boesch conjectured in 1986 that UMRGs always exist for all the pairs of n and e, and that t-optimal graphs are always λ-optimal [15]. Currently, we know that both conjectures are false (Figure 1 shows a simple counterexample of both conjectures). However, he also conjectured that t-optimal graphs are always almost regular, and thus far, there is neither formal proof nor counterexample.

This section is organized as follows. The progress on λ-optimality and t-optimality is briefly discussed in Sections 3.1 and 3.2. Some infinite families of counterexamples to the Boesch conjecture on the existence of UMRGs are presented in Section 3.3. The UMRGs known thus far are shown in Section 3.4.

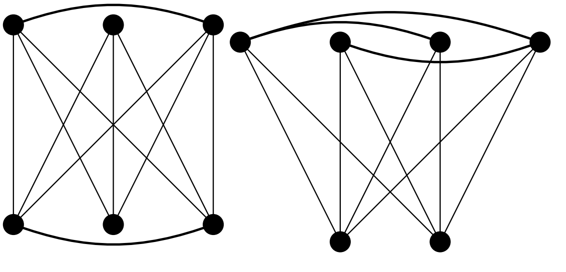

FIGURE 1 The t-optimal graph (left) is not λ-optimal (right) when $(n, e) = (6, 11)$.

3.1 | Connectivity and λ-optimality

Berge posed 14 unsolved problems in his classical book on graph theory [10]. Problem 11 challenges the readers to find the maximum connectivity and minimum diameter of an (n, e)-graph. Harary, in a simple and elegant paper, proposed a full solution [38]. Recall that the Handshaking Lemma states that $2e = \sum_i^n deg(v_i)$. Indeed, *each edge represents a handshaking between two nodes*; hence each edge contributes two units on the right-hand side of the equation. Clearly, the minimum degree, $\delta(G)$, is dominated by the average degree: $\delta(G) \leq 2e/n$. Furthermore, the edge connectivity $\lambda(G)$ cannot be greater than $\delta(G)$. Therefore, $\delta(G) \leq \lfloor 2e/n \rfloor$, and Harary constructed a special family of graphs that achieve equality, meeting the maximum edge connectivity [38]. These graphs achieve the equality $\lambda(G) = \kappa(G)$, where $\kappa(G)$ is the *node connectivity*, and since the inequality $\kappa(G) \leq \lambda(G)$ is always achieved, these graphs also achieve the maximum node connectivity [82].

If we want to construct the $2r$-regular Harary graph with n nodes $H_{2r,n}$, first locate the nodes in a regular polygon with labels 0, 1, …, $n-1$, and then connect each node i with the *closest nodes j in the polygon* such that $j = i \pm t$ for all $t \in \{1, \ldots, r\}$, where the operations are performed in modulo n. A similar construction holds for $H_{2r+1,n}$, but a matching is added linking diametrically opposite nodes (and by the Handshaking Lemma, regular graphs must have even order n in this case). Figure 2 illustrates the Harary graph $H_{3,12}$, which is also known as the Wagner graph (who considered this graph for the characterization of nonplanarity). The original work of Harary also shows a construction for general nonregular graphs [38].

In the synthesis problem, it is useful to count the edge-disconnecting sets with $\lambda(G)$ edges, that is, $m_{\lambda(G)}$. Even though finding the entire cut-vector $m(G)$ is a hard task, the number $m_{\lambda(G)}$ can be efficiently found [6]. Harary graphs achieve the maximum edge connectivity, but we also require the minimum value for m_λ for λ-optimality. This minimum was first obtained when $2e/n$ is integer and then for general pairs of n and e [8]. A *trivial cut* consists of all the incident edges of a fixed node. Observe that in a λ-regular graph $m_\lambda \geq n$, since it has at least n trivial cuts. The key concept considered by Bauer *et al.* [8] is *superconnectivity*: a λ-regular graph is superconnected if $m_\lambda = n$. Superconnected graphs are clearly λ-optimal. The authors construct generalized Harary graphs, and they show that these graphs are superconnected whenever $2e/n \geq 3$, hence λ-optimal. Finally, if $\lfloor 2e/n \rfloor = 2$, the authors consider fair subdivisions of Harary graphs. The approach is the minimization of $m_2(G)$, solving a combinatorial optimization problem exactly. A curious fact is that Wang *et al.* [79] proved that if a graph has the greatest edge connectivity $\lambda(G)$ in its class and minimizes $m_{\lambda+1}$, then it also minimizes m_λ, and it is λ-optimal. A further improvement is given by Dong *et al.* [33], where the authors construct some generalized Harary graphs that are max-λ but also min-m_i for all $i \in \{\lambda, \ldots, 2\lambda - 2\}$. Observe that a particular subset of λ-optimal graphs does *not* necessarily include a UMRG. A characterization of λ-optimal graphs is still open. The reader can find a survey on graphs with maximum connectivity in [40].

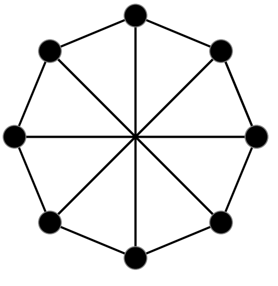

FIGURE 2 Harary Graph $H_{3,12}$, also known as the Wagner graph.

3.2 | Tree Number and t-optimality

Kirchhoff [44] studied time-invariant linear resistive circuits during the first half of the 19th century. Essentially, he wanted to find the current vector, given resistances and voltages, which defines a linear system. Surprisingly, he found a closed solution that interconnects graph theory and linear algebra forever, in a result that is known as the *matrix-tree theorem*. The result states that the tree number $t(G)$ of a graph is the magnitude of any cofactor of the Laplacian matrix $L_G = A_G - \delta_G$, with A_G being the adjacency matrix of the graph G and $\Delta_G = diag(deg(v_1), \ldots, deg(v_n))$ the diagonal matrix that stores the degrees of the nodes of G. Therefore, the tree number is efficiently found for an arbitrary fixed graph [12]. Nevertheless, the t-optimality problem is nontrivial, since the number of (n, e)-graphs (isomorphic or not) is $\binom{n(n-1)/2}{e}$, and an exhaustive search is prohibitive for general values of n and e.

To date, few structural results on t-optimal graphs have been obtained, and the general problem awaits resolution. Saccoman *et al.* [65] published a graceful book on the tree number and its applications, specifically focused on network synthesis and extended analysis for multigraphs. Kelmans *et al.* [43] and Shier [70] independently proved that the complete graph minus a matching is t-optimal. Cheng [30] established that regular complete bipartite graphs $K(m, m, \ldots, m)$ are t-optimal. In the range of sparse graphs, all the trees trivially share identical tree number $t = 1$, and elementary maximization shows that the cycle is t-optimal with $t(C_n) = n$. Recall that a θ-graph consists of two nodes linked by three disjoint paths; see Figure 3 for an illustration. Let us denote by θ_{l_1,l_2,l_3} the θ-graph whose path-lengths are l_1, l_2 and l_3. Wang and Wu [80] concluded that if $e = n + 1$, the balanced θ-graph is t-optimal. To find the tree number, just observe that we must pick and remove two edges from different paths, and $t(\theta_{l_1,l_2,l_3}) = l_1 l_2 + l_1 l_3 + l_2 l_3$, and the maximization of $t(\theta_{l_1,l_2,l_3})$ subject to $l_1 + l_2 + l_3 = e$ requires that the lengths must be balanced, i.e., $|l_i - l_j| \leq 1$ for all $i \neq j$.

The next step for $e = n + 2$ was independently solved by Tseng *et al.* [74] and Boesch *et al.* [17]. The first work formally proves the t-optimality of special subdivisions of K_4 by means of a nonlinear integer programming problem [74], and the second proves further that these subdivisions define an infinite family of UMRGs [17]. In an ambitious work, Wang [78] fully characterized all t-optimal graphs such that $e = n + 3$, and these graphs are special subdivisions of the bipartite complete graph $K_{3,3}$. It turns out that these graphs are UMRGs as well. See Figure 5 for a representation of these t-optimal graphs, which are in fact also UMRGs. Ath and Sobel [4] conjectured that the t-optimal graphs for $e = n + i$ and $i \in \{4, 5, 6, 7\}$ are special subdivisions of the Wagner, Petersen, Yutsis and Heawood graphs respectively (see Figures 2 and 4 for illustrations of these graphs). Based on computational evidence, the authors further conjectured that these graphs are UMRGs. Additional heuristics confirm that the previous graphs and the Möbius-Kantor graph are both t-optimal and λ-optimal. Formal proofs that these graphs are UMRGs are still awaiting [4].

Observe that all the known t-optimal graphs are regular or almost regular, and the corresponding conjecture posed by Boesch that t-optimal graphs are almost regular is still open. Gilbert and Myrvold [35] propose a novel algebraic technique to find t-optimal graphs when some disjoint paths or cycles are removed from the complete graph. Petingi *et al.* [53] develop algebraic techniques involving eigenvalues to find t-optimal graphs for $e \geq n(n-1)/2 - n + 2$. The cases where $e = n(n-1)/2 - n + 1$ and $e = n(n-1)/2 - n$ were also included whenever e is a multiple of 3. In a second article, Petingi *et al.* [54] found an interplay between the degree sequence of a graph and its tree number. As a consequence, the authors generalize the Cheng result, showing that the complete almost regular multipartite graphs are t-optimal. The range $n(n-1)/2 - 3n/2 < e < n(n-1)/2 - n$ is also covered; the interested reader is suggested to consult [54] for further details.

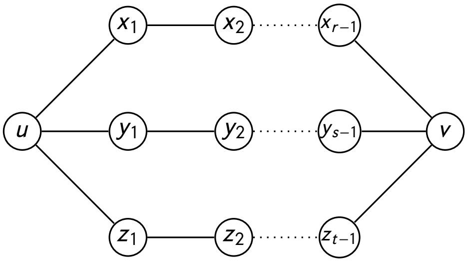

FIGURE 3 θ-graph with lengths r, s and t.

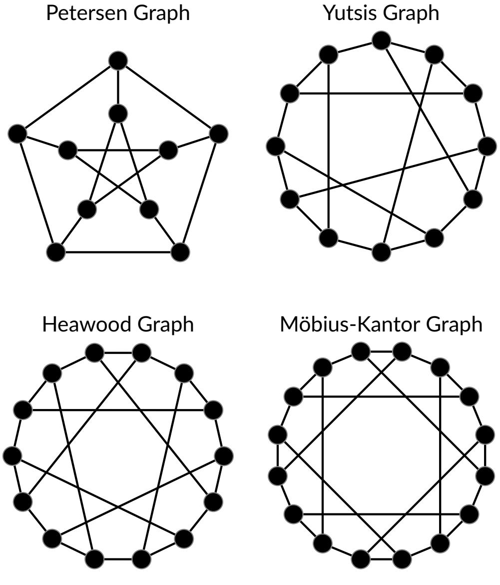

FIGURE 4 Petersen and Yutsis graphs are UMRGs. Heawood and Möbius-Kantor graphs are presumably UMRGs.

3.3 | Nonexistence of UMRGs

Infinite families of counterexamples to the Boesch conjecture on the existence of UMRGs are available in the scientific literature [27, 42, 52]. In an early work, Kelmans offered a sketch of the proof that if $n \geq 6$ is even and $e = \binom{n}{2} - \frac{n+2}{2}$, or if $n \geq 9$ is odd and $e = \binom{n}{2} - \frac{n+5}{2}$, no UMRG exists [42]. A detailed proof is given by Myrvold *et al.* [52], showing that the locally optimal graphs when ρ is close to 0 and close to 1 do not coincide. The smallest counterexample occurs when $n = 6$ and $e = 11$; see Figure 1 for an illustration of the concrete t-optimal and λ-optimal graphs. Brown *et al.* [27] covered another infinite family of counterexamples in specific pairs (n, e) such that $\binom{n}{2} - n < e < \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$. The point of departure is the t-optimality of some almost complete graphs offered by Petingi *et al.* [54]. Brown *et al.* show that even though these graphs are the most reliable for ρ sufficiently close to 1, many of them are not the most reliable for ρ close to 0. The authors also prove the nonexistence of UMRG in the extended family of multigraphs; see [27] for details.

3.4 | Existence of UMRGs

The first work that shows the existence of UMRGs was authored by Boesch *et al.* [17]. Recall that the *corank* of a connected (n, e)-graph G is $c = e - n + 1$. The authors fully characterize UMRGs with corank $c \leq 3$. Let us briefly mention the main ideas of this fundamental article [17]:

- If $c = 0$, all the tree-graphs share the same reliability polynomials, and trees are UMRGs.
- If $c = 1$, the elementary cycle C_n is UMRG. A straightforward analysis leads us to conclude that C_n is min-m_i for all i. In fact, all the (n, n)-graphs share $m_k = \binom{e}{k}$ for $k \geq 2$, and C_n is the only graph in its class such that $m_0(C_n) = m_1(C_n) = 0$.
- If $c = 2$, Bauer *et al.* [8] already constructed λ-optimal graphs, which are balanced θ-graphs. Wang and Wu [80] already found that t-optimal graphs are precisely balanced θ-graphs using combinatorial arguments. Since $m_k = \binom{e}{k}$ for all $k \geq 3$ and the members of this class, we conclude that balanced θ-graphs are UMRGs.
- If $c = 3$, the analysis is deeper. Boesch *et al.* [17] formally prove that fair node insertions of the complete graph K_4 are UMRGs. Observe that λ-optimality has been previously determined by Bauer *et al.* [8]. All the graphs belonging to this class have $m_4 = \binom{e}{4}$, and the analysis is focused on determining the t-optimality of the candidate graphs. First, they formally prove that a UMRG cannot have parallel chains (that is, induced node-disjoint paths ending in common nodes whose degrees are strictly greater than 2). The authors conclude that the UMRGs must be certain subdivisions of K_4, and finally, they find the best sequence of node insertions. Figure 5 presents the node insertions that define all the UMRGs such that $e = n + 2$, starting from K_4.

Boesch *et al.* [17] conjectured that a similar node insertion procedure starting from the bipartite complete graph $K_{3,3}$ generates all the UMRGs for corank $c = 4$. The conjecture is correct, and it was formally proven by Wang [78]. Figure 5 presents the node insertions that define UMRGs starting from $K_{3,3}$. Again, the rationale behind its proof is to show that parallel chains are not allowed, all the chains must have identical or almost identical length, and finally that the UMRGs must be certain subdivisions of $K_{3,3}$. Wang concludes that the graph class conjectured by Boesch *et al.* defines UMRGs, and further, the graphs are min-m_i.

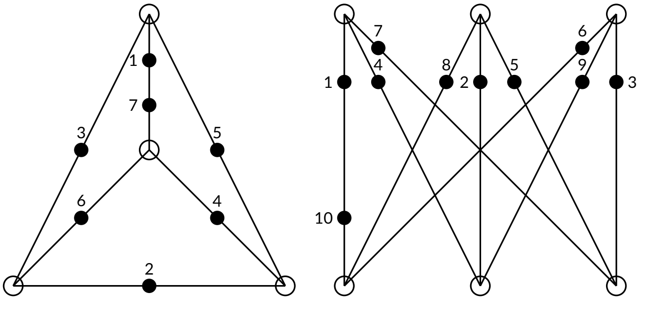

FIGURE 5 Edge-subdivisions for K_4 (left) and $K_{3,3}$ (right). The pattern is cyclic with periods 6 and 9, respectively. This results in the infinite families of UMRGs for $e = n + 2 \geq 6$ and $e = n + 3 \geq 9$.

An intermediate result by Wang is that UMRGs must be biconnected (that is, G has no cut-nodes), whenever $e \geq n$. This result has been proven in an alternative manner by Canale *et al.* in two steps [29]. First, it is proven that if H has a bridge, there exists a bridgeless graph G_1 such that $m_i(G_1) \leq m_i(H)$ for all i. Then, if G_1 is not biconnected we can find a biconnected graph G_2 such that $m_i(G_2) \leq m_i(G_1)$ for all i and conclude that a nonbiconnected graph H is uniformly less reliable than G_2. Graphs G_1 and G_2 are constructed using *shortcuts*; see Figures 6 and 7.

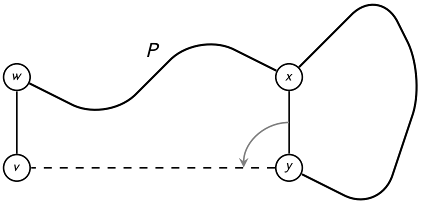

FIGURE 6 Bridgeless graph G_1. Replacing (xy) by (yv), the bridge (wv) is avoided.

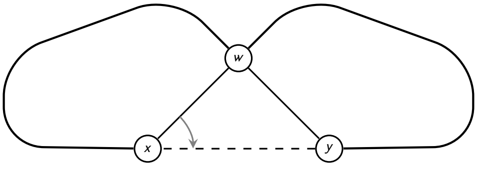

FIGURE 7 Biconnected graph G_2. Replacing (wx) by (xy), the cut-node w is avoided.

The progress in the graph classes with reduced corank is slow, and modest proofs for particular values of n and e were recently covered. Ath and Sobel [4] conjectured the *shape* of UMRGs for coranks $c \in \{5,6,7,8\}$. Computational experiments carried out by the authors suggest that some special subdivisions of the Wagner, Petersen, Yutsis and Heawood graphs are UMRGs. Additional heuristics suggest that the Cantor-Möbius graph is also a UMRG [22]. More recently, it was analytically proven that some specific cubic graphs such as the Wagner [59], Petersen [58] and Yutsis [29] graphs are UMRGs. The common methodology in these works is to prove that these graphs are in fact min-m_i for all $i \in \{0,\ldots,e\}$. Llagostera *et al.* [46] provided an analysis of UMRGs with bounded corank, restricted to the class of Hamiltonian graphs.

Boesch *et al.* [17] originally stated without mathematical proof that $K_{4,4}$ is UMRG. This claim was computationally confirmed by Myrvold, who presented a list of all UMRGs with eight nodes or fewer [50]. A mathematical proof that $K_{4,4}$ is UMRG recently appeared [28]. Another nontrivial 4-regular UMRG is $\overline{K_3 \cup C_4}$ with seven nodes [50]. See Figure 8 for the known nontrivial 4-regular UMRGs thus far. Myrvold [50] conjectured that UMRGs must have the largest girth and minimum diameter among the members of its class. Ath and Sobel [3] published a counterexample that shows, finding a couple of graphs with $n = 6$ nodes, that the minimum diameter conjecture is false. Figure 9 serves as a self-explanatory counterexample, where the UMRG is the graph on the left (the result of two node insertions to K_4). Interestingly, Ath and Sobel in the same article suggest that the largest girth conjecture is false. The authors give a candidate counterexample for the pair $(n,e) = (30,37)$; see Figure 10. Observe that graphs H_1 and H_2 have girth $g(H_1) = 9$ and $g(H_2) = 10$ respectively. If H_1 is UMRG, then the largest girth conjecture is false. Ath and Sobel assert that H_1 is UMRG, but the proof is still elusive.

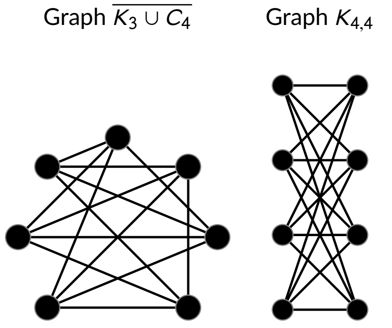

FIGURE 8 The known four regular UMRGs are K_5, K_6 minus a matching, $\overline{K_3 \cup C_4}$ (left) and $K_{4,4}$ (right).

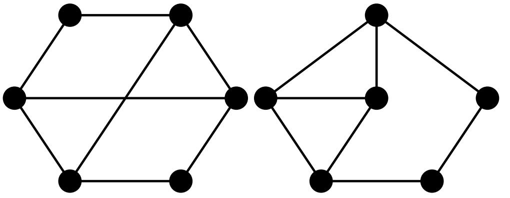

FIGURE 9 The UMRG has diameter 3 (left), but the minimum diameter is 2 (right).

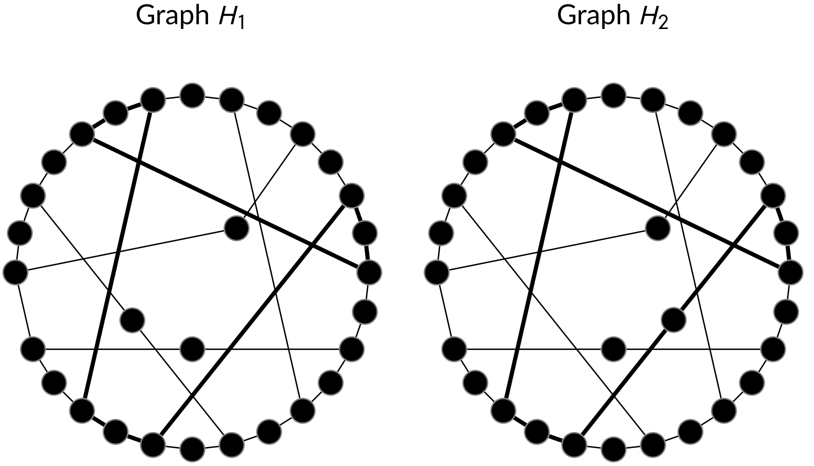

FIGURE 10 Two graphs H_1 (left) and H_2 (right) with 30 nodes, 37 edges and girth $g(H_1) = 9$ and $g(H_2) = 10$. If H_1 is UMRG, then the largest girth conjecture is false. The shortest cycles are determined with thick lines.

In the range of almost complete graphs, few UMRGs were found, covering the cases where no more than n edges are removed to the complete graph K_n [2, 42, 67]. Kelmans [42] and Satyanarayana *et al.* [67] independently found a reliability-increasing operation called *swing surgery*. This operation returns a more reliable graph under certain circumstances, and it can be combined with counting edge-disconnecting sets to find new UMRGs. Given its importance in this field, the specific transformation is explained in the following result, which is a corollary of a more general theorem due to Kelmans [42]:

Corollary 1 *Let x, y, and z be distinct nodes of a simple graph H. Further assume that:*

- $N_H(x) - \{y\} \subseteq N_H(y)$, *and*
- $z \in N_H(y) - (N_H(x) \cup \{x\})$.

If $G = H - (yz) + (xz)$, then $R_G(\rho) \geq R_H(\rho)$ for all $\rho \in [0, 1]$.

Figure 11 presents a pictorial example.

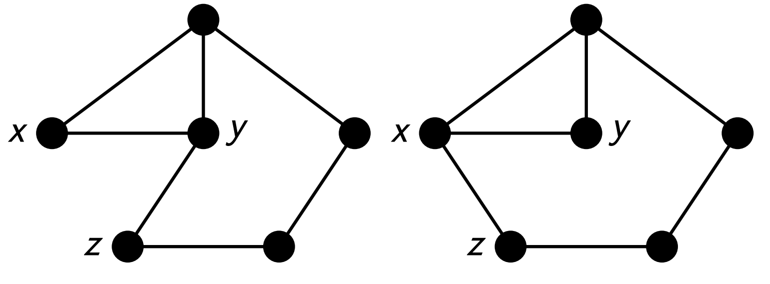

FIGURE 11 By swing surgery, a more reliable graph is obtained replacing (yz) by (xz).

An interpretation of swing surgery in the complementary graphs implies that K_n minus an arbitrary matching defines UMRGs [42]. This is an alternative topological proof for the t-optimality of K_n minus a matching, which was established by Shier using algebraic graph theory [70]. In a more recent work, Archer *et al.* [2] fully characterized UMRGs when $n \geq 5$ is odd and $e = \binom{n}{2} - \frac{n+1}{2}$ or $e = \binom{n}{2} - \frac{n+3}{2}$. The complementary graphs are a matching plus P_3 or C_3, respectively. These results serve as a complement to the cases of nonexistence provided by Brown *et al.* [27], covering the study of pairs (n, e) such that $\binom{n}{2} - n + 2 \leq e \leq \binom{n}{2}$. Figure 12 summarizes the UMRGs known thus far, and the pairs (n, e) where a UMRG does not exist in a *graph constellation* as a function of the possible pairs (n, e). The red points represent the pairs of (n, e), where it is known that a UMRG does not exist, and green points, where UMRGs exist. The straight lines include the infinite pairs where it is conjectured that UMRGs exist, using special subdivisions [4, 22]. The known UMRGs are depicted in each pair (n, e), except for complete or almost complete graphs to avoid overlaps.

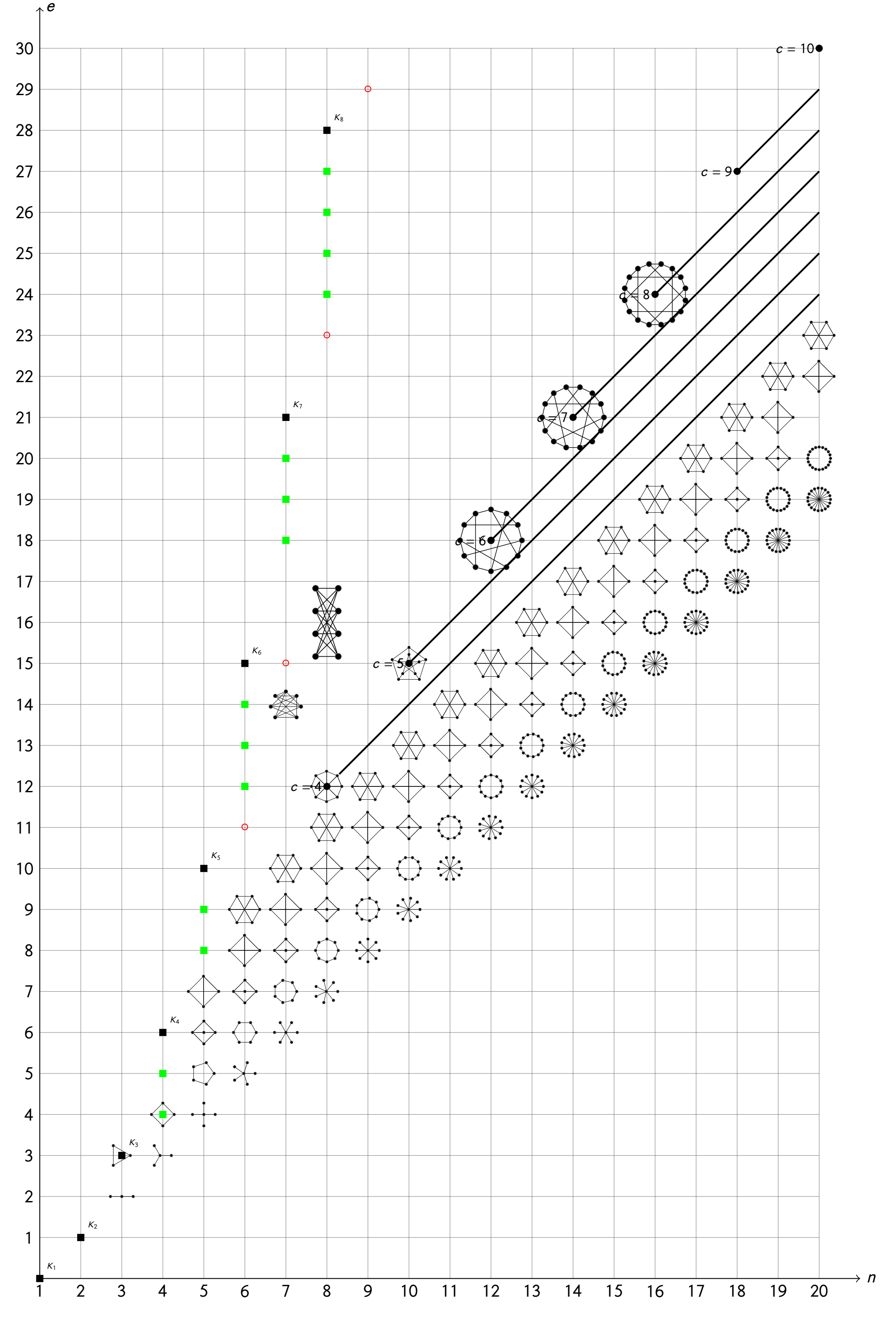

FIGURE 12 UMRGs found thus far as a function of (n, e)

4 | NODE RELIABILITY

Boesch in a beautiful work [13] describes two synthesis problems: finding locally optimal graphs for fixed values of ρ and uniformly the most reliable graphs under node failures. The problem is formally established and well motivated in terms of computer communications. The following paragraphs briefly describe these problems.

Consider a simple connected (n,e)-graph $G=(V,E)$ with perfect edges, but subject to node failures with probability ρ. The system works only if the subgraph induced by the operational nodes is connected and, further, it has at least two operational nodes. If $m_k(G)$ denotes the number of node-disconnecting sets of size k, then $m_i(G)=0$ for all $i<\kappa(G)$, where $\kappa(G)$ denotes the node connectivity of G. By the sum-rule, the unreliability polynomial is:

$$P_n(G,\rho)=\sum_{k=\kappa(G)}^{n} m_k(G)\rho^k(1-\rho)^{n-k} \tag{2}$$

The first synthesis problem posed by Boesch is to find a graph G that minimizes $P_n(G,\rho_0)$ for a fixed value $\rho_0 \in (0,1)$, among the family of (n,e)-graphs. Since the domain is finite, the existence of a solution is certain.

On the one hand, if we look for locally optimal graphs in some neighborhood of $\rho=0$, the polynomial is roughly $m_{\kappa(G)}\rho^{\kappa(G)}(1-\rho)^{n-\kappa(G)}$, and the minimization requires having the largest node connectivity $\kappa(G)$, and additionally the least number of node-disconnecting sets $m_{\kappa(G)}$. The graphs that achieve this property are called *κ-optimal graphs*. Harary graphs are κ-optimal, and general multipartite regular graphs $K(n,n,\ldots,n)$ are always κ-optimal [16]. Smith *et al.* presented a step-by-step procedure to construct κ-optimal graphs based on Harary graphs, when n/κ is a positive integer [71]. On the other hand, if ρ is close to 1, the numbers $m_n(G)=1$, $m_{n-1}(G)=n$ and $m_{n-2}(G)=\binom{n}{n-2}-e$ are identical for fixed n and e, and the invariant $m_{n-3}(G)$ makes the difference for ρ close to 1. By the inclusion-exclusion principle [62]:

$$m_{n-3}(G)=\binom{n}{3}-\sum_{i=1}^{n}\binom{deg(v_i)}{2}+2\tau(G), \tag{3}$$

with $\tau(G)$ being the number of induced triangles in G. A graph is 3-*optimal* if it minimizes the number $m_{n-3}(G)$. Clearly, locally optimal graphs in a neighborhood of $\rho=1$ must be 3-optimal. Observe that triangle-free graphs represent a good subclass to explore, given that $m_{n-3}(G)$ is increasing with $\tau(G)$. Observe that trees are in particular triangle-free graphs, and the minimization of m_{n-3} is feasible and straightforward using convex optimization. The result is that the star-graph $K_{1,n-1}$ is the only 3-optimal graph among all the trees.

The second synthesis problem is to find an (n,e)-graph that minimizes the unreliability polynomial $P_n(G,\rho)$ in a uniform sense, for all $\rho\in[0,1]$. Boesch shows that 3-optimal and κ-optimal graphs do not coincide when $n=e\geq 5$, and concludes that UMRGs do not always exist under this node reliability setting [13]. In fact, the largest node connectivity when $n=e$ is $\kappa(G)=2$, and the elementary cycles are the only κ-optimal graphs. If $n=3$ or $n=4$, it is straightforward to check that the elementary cycles C_3 and C_4 are in fact UMRG. However, if $n=e\geq 5$, the star graph plus one edge are 3-optimal graphs when $n=e\geq 5$, and UMRGs do not exist. Stivaros in his doctoral thesis [72] proved that the star graphs are not only κ-optimal but also UMRGs. Furthermore, Stivaros showed that the complete graph minus an arbitrary matching is UMRG as well and provided a discussion of 3-optimality, κ-optimality and heterogeneous node failures. In particular, all 3-optimal triangle-free and C_5-free graphs are bipartite. Consequently, most results are concerned with the search for complete bipartite UMRGs or their nonexistence for special pairs of n and e.

Goldshmidt *et al.* [36] prove that almost regular complete k-partite graphs are always UMRGs. A smart lower bound for the numbers $m_G(k)$ is first obtained, and finally, the authors deduce that this distinguished subclass has the least number of node-disconnecting sets. The authors attribute this result to Bermond, even though the full proof is introduced by them. Their first extension is to prove that $K_{b,b+2}$ is UMRG on its class, and it is the *most optimistic result*, in the sense that $K_{b,b+i}$ is never UMRG when $i > 2$, and further, that nonregular complete multipartite graphs whose parts have either b or $b+2$ nodes are never UMRG. Finally, the nonexistence of UMRGs for some pairs (n, e) such that $e < n^2/4$ is established. Using Stivaros' result that 3-optimal (triangle-free and C_5-free) graphs are bipartite, the authors modify regular bipartite graphs with addition/removal of some edges, finding graphs with good performance in terms of 3-optimality. Then, they show that κ-optimal graphs have greater values of m_{n-3}, hence κ-optimal graphs are never 3-optimal [36]. Later, Liu *et al.* [45] proved that $K_{b,b+1,b+2}$ is UMRG in its class, but $K_{b,b+1,b+i}$ is never UMRG if $i > 2$. The first result involves elementary combinatorics, and the second is just to observe that the node connectivity of the graphs $K_{b,b+1,b+i}$ is not maximum when $i > 2$. A further generalization is carried out by Yu *et al.* [84], where the authors prove that $K_{b,b+1,b+1,\ldots,b+1,b+2}$ is UMRG in its class, but $K_{b,b+1,b+1,\ldots,b+1,b+i}$ for $i > 2$ is never UMRG.

5 | ADDITIONAL RELIABILITY MODELS

In this section we consider two-terminal reliability models under either node or edge failures, uniformly least reliable graphs and some open problems dealing with multigraphs.

5.1 | Two-Terminal Reliability Model under Edge Failures

The corresponding synthesis problem under the two-terminal reliability setting was recently launched by Bertrand *et al.* [11]. Consider two terminals $s = v_1$ and $t = v_n$ and the remaining nonterminal nodes $v_2, \ldots, v_{n-1}$. Let $N_k(G)$ be the number of $s-t$ pathsets with precisely k edges. If p denotes the operational probability of the individual edges, then the *two-terminal reliability* $R_2(G; p)$ is:

$$R_2(G;p) = \sum_{k=1}^{e} N_k(G) p^k (1-p)^{e-k}. \tag{4}$$

Intuitively, we should greedily pick the shortest $s-t$ paths first, including edge st, then the greatest number of 2-paths N_2, 3-paths N_3, and so on. This intuition is first captured by the following result, which is a slight modification of the reasoning posed by Brown *et al.* [27] for the all-terminal reliability setting:

Lemma 2

- *If $N_i(G) = N_i(H)$ for $1 \leq i < k$ but $N_k(G) > N_k(H)$, G is more reliable than H for p close to* 0.
- *If $N_i(G) = N_i(H)$ for $l < i \leq e$ but $N_l(G) > N_l(H)$, G is more reliable than H for p close to* 1.

Following the previous intuition, Bertrand *et al.* [11] prove that a distinguished family of graphs is the most reliable when p is close to 0:

Definition Let $n \geq 3$ and $0 \leq r \leq n-3$. The graph $A_{n,r}$ has distinguished nodes $s = v_1$, $t = v_n$ and edges st, sv_i, v_it for all $2 \leq i \leq n-1$, and also v_2v_j for all $3 \leq j \leq r+2$.

See Figure 13 for a representation of $A_{n,r}$. This greedy construction of disjoint 2-paths is in fact most reliable when p is close to 0:

Proposition 3 *If $n \geq 4$ and $5 \leq e \leq 2n-3$, the unique most reliable graph for p close to 0 is $A_{\lfloor \frac{e+2}{2} \rfloor, (e+1) \, mod \, 2}$ together with $n - \lfloor \frac{e+2}{2} \rfloor$ isolated nodes.*

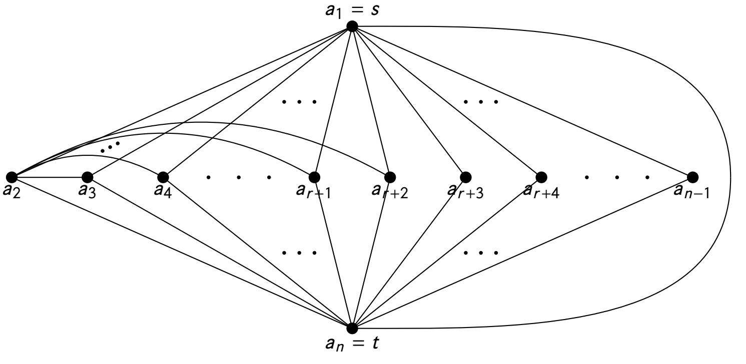

FIGURE 13 The graph $A_{n,r}$

The authors find a closed formula for the two-terminal reliability $R_2(A_{n,r}; p)$ and consider a structural modification $A'_{n,r} \neq A_{n,r}$ such that $R_2(A'_{n,r}; 1/2) > R_2(A_{n,r}; 1/2)$. The nonexistence of UMRGs is concluded in this way for $n \geq 11$ and $20 \leq e \leq 3n-9$.

Bertrand *et al.* [11] studied almost complete graphs as well. The most reliable graphs for p close to 0 and 1 do not coincide when $\binom{n}{2} - \lfloor \frac{n-2}{2} \rfloor \leq e \leq \binom{n}{2} - 2$. Clearly, the complete graph K_n is the UMRG when $e = \binom{n}{2}$. Finally, if $e = \binom{n}{2} - 1$ we should remove either st, sv_2 or v_2v_3 to find only three nonisomorphic graphs. Our intuition suggests that st should never be removed, and the authors formally prove that the UMRG is $K_n - \{v_2v_3\}$. The reader is recommended to consult [11] for further details.

This research line was subsequently followed by Xie *et al.* [83] in a recently published work. Table 1 summarizes the progress in the synthesis problem under edge failures in the two-terminal reliability setting.

(n, e)-graphs	UMRGs
$n \geq 11$ and $20 \leq e \leq 3n-9$	Nonexistence [11]
$n \geq 11$ and $3n-8 \leq e \leq 3n-6$	Unknown
$n \geq 6$ and $3n-6 < e \leq \binom{n}{2} - 2$	Nonexistence [83]
$n \geq 4$ and $e \geq \binom{n}{2} - 1$	Existence [11]

TABLE 1 Summary of UMRGs for two-terminal/edge failure model, as a function of n and e.

5.2 | Two-Terminal Reliability Model under Node Failures

If there are no further restrictions, the two-terminal reliability model under node failures is trivial: just connect the terminals with a perfect edge, and the reliability is the maximum. Observe that we assume that only the nonterminal nodes fail, and the edges are perfect.

If we force the distance between the terminals to be $d(s,t) \geq 2$, the intuition is that we must pick as many disjoint 2-paths as possible between s and t. In fact, observe that the graph $B_{n-2} = K_{2,n-2}$ with precisely $n-2$ disjoint 2-paths between the terminal nodes has the best reliability, even when compared with an arbitrary simple graph G:

$$R_{B_{n-2}}(\rho) = 1 - \rho^{n-2} \geq R_G(\rho), \tag{5}$$

where ρ denotes the node failure probability. The rationale behind this inequality is that B_{n-2} works if and only if at least some node works. Nothing better can be achieved unless edge st is included, but in this case, we would have $d(s,t) = 1$. Brown *et al.* [23] prove that the previous greedy construction (i.e., an iterative addition of disjoint 2-paths) defines UMRGs with the constraint that $d(s,t) \geq 2$. The key concept is a *weak node*: v is weak if it is connected to precisely one terminal node $z \in \{s,t\}$, and it has at least some other neighbor $x \notin N(z)$. If the edges $E_v = \{vx : x \notin N(z)\}$ are correspondingly replaced by $E_z = \{zx : vx \in E_v\}$, the resultipng graph is more reliable [23]. This reliability-increasing transformation is an adequate variation of *swing surgery*, suitable for the two-terminal node reliability model. Observe that $B_r = K_{2,r}$ does not have weak nodes, and define UMRGs for $d(s,t) = 2$.

A shocking result is that there are no UMRGs if we force $d(s,t) \geq 3$, at least when the number of edges e is not sufficiently large. Under this model, the most reliable graphs must have the largest node connectivity in a neighborhood of $\rho = 0$. By Menger's theorem [39], the number of node-disjoint 3-paths between s and t must be maximized. However, in a neighborhood of $\rho = 1$ nodes fail frequently, and the number of 3-paths (disjoint or not) must be maximized instead. These extremal problems are clearly different, and the locally optimal graphs do not coincide when e is small [23]:

Theorem 4 *There exists UMRGs subject to $d(s,t) \geq 3$ if and only if:*

- $e \leq 8$;
- $n = 7$ *and* $e = 12$, *or*
- $e \geq \lfloor \frac{(n+1)^2}{4} \rfloor$.

The particular cases where $e \leq 8$ or $(n,e) = (7,12)$ can be computationally studied (the reliability polynomial can be obtained and compared). The authors of [23] consider H_n to be the two-terminal graph with $n+2$ nodes, where the terminal nodes are fully linked with the separate partitions of a bipartite complete graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. They show that any graph containing H_n as a special subgraph is UMRG, sharing an identical reliability polynomial provided that $d(s,t) \geq 3$.

Interestingly, they also conjectured that no UMRG exists if we force $d(s,t) = d \geq 4$, whenever $n \geq 2(d-1)$ and the size e satisfies the following relation:

$$2d \leq e \leq (d-2)\left\lfloor \frac{n}{d-1} \right\rfloor^2 + \left\lfloor \frac{n}{d-1} \right\rfloor - 1. \tag{6}$$

A proof strategy for this conjecture is suggested, where again, the locally optimal graphs for ρ close to 0, or 1, are presumably disjoint. The interested reader can find full details in [23].

5.3 | Uniformly Least Reliable Graphs (ULRGs)

An (n, e)-graph G is a ULRG if its reliability is uniformly the worst; formally, $U_G(\rho) \geq U_H(\rho)$ for all (n, e)-graphs H and $\rho \in [0, 1]$. Since the all-terminal reliability evaluation is a $\#P$-complete problem, ULRGs serve in practice to build universal reliability bounds, together with UMRGs. If p denotes the operational probability of the individual edges, the reliability polynomial can be expressed in its general form [19]:

$$R_G(p) = t(G)p^{n-1} + \sum_{k=n}^{e-1}(-1)^{k-(n-1)} f_k(G)p^k + (-1)^{e-n+1} D(G)p^e, \tag{7}$$

where $t(G)$ is the tree number and $D(G)$ is the *domination* of graph G. Recall that the domination in reliability theory is the magnitude of the difference between odd-formations and even-formations (and a formation is a set of minimally operational subgraphs whose union equals the whole graph; the reader can find further details in [1]). As in the case of UMRG, both invariants $t(G)$ and $D(G)$ must be minimal in a ULRG [19]. Even though the determination of t-optimal graphs is still not well understood, Bogdanowicz [21] fully characterized all the graphs with the least tree number in a strict sense. The key is, first, to observe that an iterative application of *swing surgery in reverse* determines a partial order, and it is a reliability-decreasing graph transformation. The graph set that consists of minimal elements (or invariant graphs under this operation) is precisely the threshold graphs [20, 67].

The exact determination of the graph domination $D(G)$ is $\#P$-complete; see [6] for a proof and [19] for its combinatorial interpretation. Boesch *et al.* [18] proved that threshold graphs attain the minimum domination $D(G)$, but this minimization is also achieved by other graphs. Specifically, $D(G)$ is minimum if and only if G is either a threshold graph or it consists of m blocks, such that one block is K_{n-m}, the second is K_3 and the other blocks are single edges. Petingi *et al.* [55] studied ULRGs in an almost complete case when $e > (n-1)(n-2)/2$. The authors defined a *balloon graph* as a complete graph K_{n-1} with a single cone of degree $k : 0 < k \leq n$. After a careful analysis of reverse swing surgery in the complementary graphs, they concluded that balloon graphs are ULRGs. Furthermore, these graphs accept an exact reliability evaluation, which provides universal reliability bounds [26].

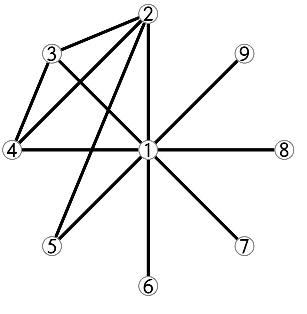

FIGURE 14 Generalized Balloon Graph $G_{n,e}$ with $n = 9$ nodes, $e = 12$ edges and $i = 12 - 9 + 1 = 4$ edges added to $K_{1,8}$. These edges are $(2, 3)$, $(2, 4)$, $(3, 4)$ and $(2, 5)$.

A recent study presents a generalization of balloon graphs $G_{n,e}$ for all pairs of n and e [61]. These graphs are inductively defined as the smallest set such that $G_{n,n-1} = K_{1,n-1}$, and $G_{n,e+1}$ adds an edge to $G_{n,e}$, maximizing the resulting number of bridges. An example is depicted in Figure 14. The author proves that $G_{n,e}$ is ULRG whenever $e \leq n+3$. A key element is to observe that $f_k(G)$ is invariant under bridge contractions. Then, bridges are removed, and the blocks of the resulting graph are studied using the Whitney characterization theorem for biconnected graphs [81].

As the main conclusions, the candidates for ULRGs are fully characterized. This is in strong contrast with UMRGs. These graphs, if they exist, are in general not unique. The existence is guaranteed thus far for sparse graphs with reduced corank ($e \leq n+3$) or almost complete graphs ($e \geq (n-1)(n-2)/2$). The main conjecture is that generalized balloon graphs are ULRGs.

5.4 | Reliability in Multigraphs

A brief combinatorial argument shows that if we consider all the (n, e)-multigraphs with $e = \binom{n}{2}$ edges (parallel edges are allowed), then the t-optimal graph is K_n, which is the unique simple graph belonging to this class [43]. In fact, we should maximize the product of the positive eigenvalues λ_i of the Laplacian matrix, subject to the constraint $2e = \sum_i \lambda_i$. This maximization is achieved when the nonnegative eigenvalues are identical, precisely in the complete graph K_n. Two related questions arise from this fact:

1. Is t-optimality preserved in the extended family of multigraphs?
2. Is the uniformly optimal reliability preserved as well?

These questions open the door to decide the construction of multiple edges if profitable. It is an easy exercise to find examples where the answer is negative under heterogeneous edge failures. However, the matter is nontrivial under identical and independent edge failures. Gross and Saccoman [37] formally proved that the answer to both problems is affirmative when $e \leq n+2$ and conjectured that the subdivisions of $K_{3,3}$ are UMRGs in multigraphs. Using a laddering domination technique (showing that graphs with corank i are uniformly least reliable than some graph with corank $i+1$ for all $i \in \{0, 1, 2, 3\}$ in the extended class of multigraphs), a recent work confirms that the Gross-Saccoman conjecture is true [60]. This laddering domination methodology a priori does not scale with the corank, and the answer is yet unknown for all the remaining cases where $e \geq n+4$.

6 | CONCLUDING REMARKS

The study of the uniformly most reliable graphs (UMRGs) serves as a guide for network design and decision-making. During recent decades, abundant conjectures have been proposed, but most of them are still unsolved. The level of abstraction in the problems invites mathematicians and computer scientists to develop and/or combine algebraic graph theory, combinatorics, probability and calculus, matroid theory, combinatorial optimization and algorithmic complexity, among many other fields, simultaneously. The search for symmetry and beauty is intriguing, and the existence, uniqueness and construction of UMRGs frequently appear to be the main mathematical questions.

Boesch, in his foundational article [15], claimed that UMRGs always exist, but infinite families of counterexamples were provided by different authors. If a graph minimizes the number of edge-disconnecting sets, it is UMRG. The converse is a major conjecture in this field. If affirmative, there exists an optimal network design from both probabilistic (reliability-oriented) or deterministic (connectivity-oriented) points of view. Finding the (n, e)-graphs with the greatest tree number, or t-optimal graphs, is a challenging task, which is understood only for sparse or almost complete graphs. The (n, e)-graphs with the greatest edge connectivity λ and smallest m_λ are not structurally characterized, and particular λ-optimal constructions consider Harary graphs and generalizations. The existence and construction of UMRGs is still awaiting for the range where $n \geq 9$ and $n+4 \leq e \leq \binom{n}{2} - n$. In the case of a sparse graph where $e = n+i$ for $i \in \{4, 5, 6, 7\}$, computational experiments suggest that UMRGs are special subdivisions of Wagner, Petersen, Yutsis and Heawood graphs. These conjectures are trends for future work. A common method to address these problems is to count the number of edge-disconnecting sets and adequately partition the class of (n, e)-graphs. On the other hand, the study of almost complete graphs combines reliability-improving transformations such as swing surgery with smart comparison techniques.

In the field of UMRGs under node failures, only a few cases have been covered thus far. In an analogy with the edge reliability setting, UMRGs must be both 3-optimal and κ-optimal. For example, cycles are κ-optimal but not 3-optimal, and UMRGs do not exist in this setting when $e = n$. By the inclusion-exclusion principle, triangle-free graphs are preferred in terms of 3-optimality, and κ-optimal graphs must be regular whenever possible. Regular multipartite graphs satisfy the previous conditions. In fact, all known UMRGs under node failures are bipartite or multipartite graphs, and the existence of UMRGs is still awaiting further research.

Two-terminal reliability models were recently explored under either node or edge failures in separate articles. When edge failures are considered, UMRGs do not exist when $n \geq 11$ and $20 \leq e \leq 3n-9$ or when $n \geq 6$ and $3n-6 \leq e \leq \binom{n}{2} - 2$. If we remove a single edge between nonterminal nodes in the complete graph K_n, a UMRG is obtained. The range of (n, e)-pairs such that $n \geq 11$ and $3n-8 \leq e \leq 3n-6$ is still unknown, and it is an attractive topic for future research, since it covers the full pairs (n, e), except for small values of n, which accept an exhaustive computational analysis. The two-terminal problem is trivial under imperfect nonterminal nodes and perfect edges: just connect the terminals with a perfect edge, and the reliability is the maximum. The problem is interesting if we force a minimum distance between the terminals. A greedy construction adding as many disjoint 2-paths as possible is optimum if the distance is $d = 2$. Curiously enough, if $d \geq 3$ UMRGs exist in the range of dense graphs only, or for particular pairs; see Theorem 4. A challenging conjecture of nonexistence was given for $d \geq 4$.

Universal reliability bounds are available if we can derive uniformly least reliable graphs (ULRGs). Balloon graphs are almost complete ULRGs. It is conjectured that generalized balloon graphs are ULRGs. It is interesting to observe that if ULRGs exist, they are fully characterized, in strong contrast with UMRGs. Partial answers were given, and we know that generalized balloon graphs are ULRG when $e \leq n+3$. Another attractive question from an operational point of view is to consider the extended class of multigraphs, opening the door to interconnect sites with multiple parallel links. Are UMRGs still optimum in this extended class? An affirmative answer was partially given when $e \leq n+3$.

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