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Networks

A Simple Proof of the Gross-Saccoman Multigraph Conjecture

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An enigmatic conjecture in network synthesis asserts that the the uniformly most reliable multigraphs are simple. Daniel Gross and John Saccoman proved in 1998 that the answer is affirmative whenever $m \leq n+2$, where n and m is the respective number of nodes and edges of the multigraphs. They conjectured that the optimality is also achieved by simple graphs when $m = n+3$. A proof for this conjecture recently appeared.

In this article we provide a unified short proof for the previous cases where $m \leq n+3$. Our proof strategy holds whenever the most reliable simple graphs satisfy the self similarity property. As a consequence, it could be used to study the general multigraph conjecture for larger graph classes.

KEYWORDS
Network Reliability, Gross-Saccoman multigraph conjecture, Graph Theory, Uniformly Most Reliable Graph, Self Similarity Property, Multigraph.

1 | BACKGROUND

A simple graph $G = (V, E)$ is a nonempty node-set V equipped with an edge-set E that consists of elements $e = (x y)$ where $x \neq y \in V$. In a multigraph the edges can be repeated and the edge-set E is a multi-set. In a pseudograph E may have repeated edges but also loops that consist of elements $e = (x x)$ for some node $x \in V$. Multiple loops are allowed. The reader is invited to consult the book authored by Frank Harary for the graph-theoretic terminology [5].

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Denote $\Omega_s(n, m)$, $\Omega(n, m)$ and $\Omega_p(n, m)$ the collection of all the simple graphs, multigraphs and pseudographs with n nodes and m edges, respectively. Consider a pseudograph $G = (V, E)$ such that $G \in \Omega_p(n, m)$ whose nodes do not fail but its edges have independent failure probability $\rho \in [0, 1]$. The *reliability* of G is the probability that upon failures of edges the resulting random pseudograph remains connected, and it is denoted $R_G(\rho)$. For convenience we deal with the *unreliability* $U_G(\rho) = 1 - R_G(\rho)$. An *edge-disconnecting set* of G is a subset $U \subseteq E$ such that $G - U$ is not connected. Let $m_k(G)$ be the number of edge-disconnecting sets of G with size $k \in \{0, \ldots, m\}$. By the sum-rule, the unreliability satisfies the following identity,

$$U_G(\rho) = \sum_{k=0}^{m} m_k(G)\rho^k(1-\rho)^{m-k}. \tag{1}$$

Definition Consider a pair of positive integers n and m and let $\mathcal{F}$ be a subset of $\Omega_p(n, m)$.

- A pseudograph $G \in \mathcal{F}$ is *uniformly most reliable in* $\mathcal{F}$ if $U_G(\rho) \leq U_H(\rho)$ for all $H \in \mathcal{F}$ and all $\rho \in [0, 1]$.
- A pseudograph G is *stronger* than H if $m_k(G) \leq m_k(H)$ for all $k \in \{0, \ldots, m\}$.
- A pseudograph G is the *strongest* in the set $\mathcal{F}$ if $m_k(G) \leq m_k(H)$ for all $k \in \{0, \ldots, m\}$ and all $H \in \mathcal{F}$.

From Equation (1) it is clear that the strongest graph G in $\mathcal{F}$ is uniformly most reliable, for all $\mathcal{F} \subseteq \Omega_p(n, m)$. Frank Boesch conjectured that the uniformly most reliable graph in $\Omega_s(n, m)$ is the strongest in its class, and this is a major open problem in this field [2]. Another interesting problem is to know whether the uniformly most reliable graph G in $\Omega_s(n, e)$ is also uniformly most reliable in the extended set $\Omega(n, e)$. Daniel Gross and John Saccoman [4] proved that the answer is affirmative when $m \leq n + 2$, and they conjectured that the result holds when $m = n + 3$. A recent proof of the Gross-Saccoman *conjecture* (now a theorem) recently appeared [6]. Both proofs are long and involved. In fact, both proofs propose some adequate partition $\mathcal{F}_1, \ldots, \mathcal{F}_r$ of the set $\Omega(n, m)$. Then, the authors find the strongest graph G_i for each subclass $\mathcal{F}_i$. Finally, they compare the strongest members $G_1, \ldots, G_r$ to conclude that a simple graph is the strongest in $\Omega(n, m)$.

In this work we give a unified simple proof of the multigraph conjecture for all the pairs of n and m such that $m \leq n + 3$. The proof is short, and the key is to combine deletion-contraction formula with the self similarity property [6].

Definition A graph-sequence $\{G_n\}_{n \geq n_0}$ satisfies the *self similarity property* if G_{n+1} is a node insertion of G_n, $\forall n \geq n_0$.

The following concepts are well-known for simple graphs, but we will extend some of them for pseudographs. If $G = (V, E) \in \Omega_p(n, m)$, a node insertion in the edge (xy) produces a new pseudograph $G' \in \Omega_p(n + 1, m + 1)$ such that the edge (xy) is replaced by two edges (xz) and (zy), where z is a new node belonging to G'. The deletion of an edge e is $G - e = (V, E - e)$. If $e = (xy)$ and $x \neq y$, the *edge-contraction* $G * e$ identifies the endpoints x and y of e, and the edge e is deleted. It is important to observe that in an edge-contraction both the number of nodes and edges drops down a unit, and if the nodes x and y share multiple edges $e_1, \ldots, e_s$ and we contract e_1, the new node has $s - 1$ loops after the edge-contraction.

Frank Boesch observed that the following *deletion-contraction* identity holds for multigraphs [2]:

$$m_{k+1}(G) = m_k(G - e) + m_{k+1}(G * e) \tag{2}$$

Using our definition of edge-contraction and the sum-rule, the identity also holds for pseudographs.

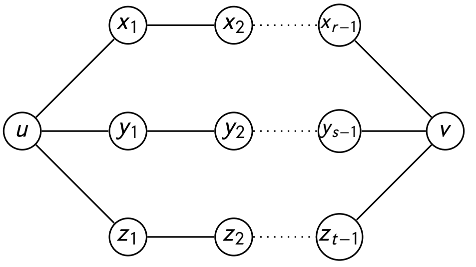

FIGURE 1 θ-graph with lengths r, s and t. A θ-graph is balanced if $|r-s| \le 1$, $|r-t| \le 1$ and $|s-t| \le 1$.

The strongest members in the classes $\Omega_s(n,m)$ are fully known when $m \le n+3$; see [3, 7]. Boesch *et al.* [3] proved that the so-called balanced θ-graphs and special node insertions of K_4 are the strongest in the respective sets $\Omega_s(n,n+1)$ and $\Omega_s(n,n+2)$, and Wang [7] proved that certain node insertions of $K_{3,3}$ define the strongest graphs in $\Omega_s(n,n+3)$. See Figures 1 and 2 for illustrations of these graphs. We denote P_n, C_n and θ_n the n-paths, n-cycle and balanced θ-graphs with n nodes, which are the strongest in their respective classes of simple graphs. Further, denote $K_u(n)$ and $G_u(n)$ the strongest members in the respective classes $\Omega_s(n,n+2)$ and $\Omega_s(n,n+3)$. The following remark is our point of departure:

Remark The self-similarity property holds for the strongest graph sequences $\{P_n\}_{n\ge2}$, $\{C_n\}_{n\ge3}$, $\{\theta_n\}_{n\ge4}$, $\{K_u(n)\}_{n\ge4}$ and $\{G_u(n)\}_{n\ge6}$.

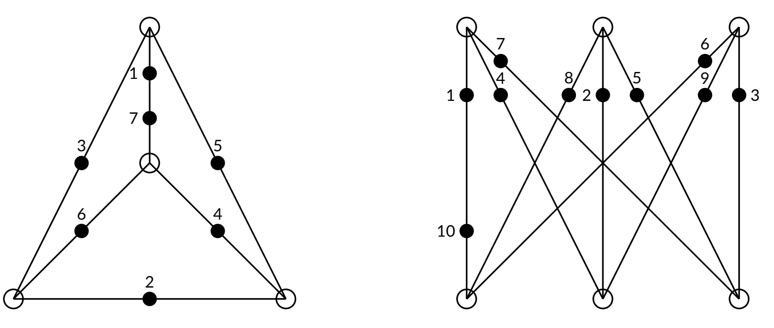

FIGURE 2 Graphs $K_u(n)$ (left) and $G_u(n)$ (right). The black nodes represent the node insertions and the labels represent the order of the respective insertions. The node insertion process is periodic with periods 6 and 9 for the respective graphs $K_u(n)$ and $G_u(n)$. The examples illustrate $K_u(11)$ and $G_u(16)$.

2 | MAIN RESULT

For each integer i such that $i \in \{0,1,2,3,4\}$, denote $X(n,i)$ the strongest graph in the class of simple graphs $\Omega^i = \Omega_s(n,n-1+i)$. We already know that $X(n,i)$ is P_n, C_n, θ_n, $K_u(n)$ and $G_u(n)$ for the respective values of $i \in \{0,1,2,3,4\}$. It is clear that $X(n,0) = P_n$ is the strongest in the class of pseudographs $\Omega_p(n,n-1)$, since the only connected graphs in $\Omega_p(n,n-1)$ are trees, and they all satisfy $m_0 = 0$ and $m_k = \binom{n-1}{k}$ for all $k \ge 1$.

From Remark 1, the sequence $\{X(n,i)\}_{n\geq n_0}$ satisfies the self similarity property for some positive integer n_0, and $X(n,i)$ is obtained by a node insertion z_n into the simple graph $X(n-1,i)$. We know that z_n is incident to precisely two edges in the resulting graph $X(n,i)$. Let us pick one of them, namely, $e_n \in X(n,i)$. Let us further define the multigraph $Y(n,i+1) = X(n,i) \cup \{e_n\} \in \Omega(n,n+i)$, that is, the graph $X(n,i)$ but with the *double-edge* e_n. Observe that $Y(n,i+1) - e_n = X(n,i)$, and $Y(n,i+1) * e_n = X(n-1,i) \cup \{l\}$, where l is a loop.

The following lemma was proved in [6], and it is included here for the sake of completeness:

Lemma 1 *If G is stronger than H in $\Omega_p(n,e)$ then $G \cup \{\ell\}$ is stronger than $H \cup \{\ell\}$ in $\Omega_p(n,e+1)$, where $\ell = (vv)$ is an arbitrary loop.*

Proof A loop ℓ can either appear in an edge-cut or not. Since G is stronger than H, then

$$m_{k+1}(G \cup \{\ell\}) = m_k(G) + m_{k+1}(G) \leq m_k(H) + m_{k+1}(H) = m_{k+1}(H \cup \{\ell\}),$$

for all $k \geq 0$. The connectedness is not modified with loops, and $m_0(G \cup \{\ell\}) = m_0(G) \leq m_0(H) = m_0(H \cup \{\ell\})$, so $G \cup \{\ell\}$ is stronger than $H \cup \{\ell\}$. □

Lemma 2 *$X(n,i)$ is stronger than $Y(n,i)$, for all $i \in \{1,2,3,4\}$.*

Proof The cases $i \in \{1,2\}$ are straightforward:

- $X(n,1) = C_n$ is stronger than $P_n \cup e_n$, since $m_1(C_n) = 0 \leq m_1(P_n \cup e_n)$, and $m_k(C_n) = m_k(P_n \cup e_n) = \binom{n}{k}, \forall k \geq 2$.
- $X(n,2) = \theta_n$ is stronger than $C_n \cup e_n$. In fact, the latter is also a θ-graph, but it is unbalanced.

Finally, if $i \in \{3,4\}$ we consider the simple graphs $G(n,3) = Y(n,3) - e_n + f$ and $G(n,4) = Y(n,4) - e_n + g$ depicted in Figures 3 and 4. It is a simple exercise in combinatorics to check that $G(n,3)$ and $G(n,4)$ are stronger than $Y(n,3)$ and $Y(n,4)$, respectively. The result follows, since the relation of stronger graphs is transitive. □

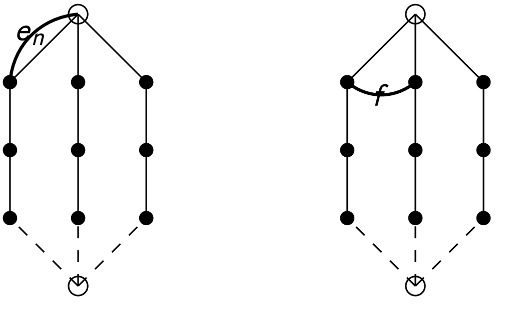

FIGURE 3 Graphs $Y(n,3)$ (left) and $G(n,3) = Y(n,3) - e_n + f$ (right).

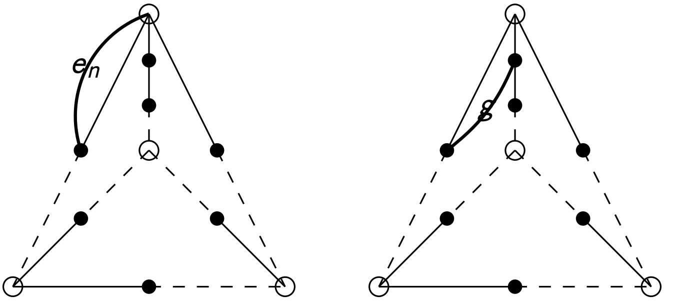

FIGURE 4 Graphs $Y(n,4)$ (left) and $G(n,4) = Y(n,4) - e_n + g$ (right)

Lemma 3 *The graph $Y(n, i+1)$ is the strongest in $\mathcal{F}^i = \Omega_p(n, n+i) - \Omega_s(n, n+i)$ for all $i \in \{0,1,2,3,4\}$.*

Proof The proof follows by induction on $i \in \{0,1,2,3,4\}$.

- Base-step: if $i = 0$ then the only connected non-simple pseudographs are the trees with precisely one repeated edge, and these graphs satisfy $m_0 = 0$, $m_1 = n + i - 2$ and $m_j = \binom{n+i}{j}$ for all $j \geq 2$. In particular, $Y(n,1)$ is the strongest in $\mathcal{F}^0$, and the base-step holds.
- Induction-step: assume that $Y(n,i)$ is the strongest in $\mathcal{F}^i$ for any fixed $i \in \{0,1,2,3\}$. Let G be an arbitrary pseudograph in $\mathcal{F}^{i+1}$. Then G either has a loop or not. Consider both cases separately:
  - If $G = G' \cup \{l'\}$ for some loop l' then

$$\begin{aligned} m_{k+1}(Y(n,i+1)) &= m_k(X(n,i)) + m_{k+1}(X(n-1,i) \cup \{l\}) && (3)\\ &\leq m_k(Y(n,i)) + m_{k+1}(Y(n-1,i) \cup \{l\}) && (4)\\ &\leq m_k(G) + m_{k+1}(G' \cup \{l'\}) && (5)\\ &= m_{k+1}(G), && (6) \end{aligned}$$

    where (3) is the deletion-contraction formula for the repeated edge in $Y(n, i+1)$, the inequality (4) uses that $X(n,i)$ is stronger than $Y(n,i)$ (Lemma 2) and Lemma 1 for loops, the inequality (5) uses the inductive hypothesis and Lemma 1 for loops and the identity (6) is the sum-rule, where l' either appears or not in an edge-cut.
  - If G is loopless in $\mathcal{F}^{i+1}$ then G must have some repeated edge f. Therefore $G = G' \cup \{f\}$ and $G * f$ has at least some loop denoted l'. A similar chain of inequalities holds in this case, where the key is to observe that $G * f$ has the loop l' and $G * f - \{l'\} \in \mathcal{F}^i$:

$$\begin{aligned} m_{k+1}(Y(n,i+1)) &= m_k(X(n,i)) + m_{k+1}(X(n-1,i) \cup \{l\}) && (7)\\ &\leq m_k(Y(n,i)) + m_{k+1}(Y(n-1,i) \cup \{l\}) && (8)\\ &\leq m_k(G') + m_{k+1}((G * f - \{l'\}) \cup \{l'\}) && (9)\\ &= m_{k+1}(G), && (10) \end{aligned}$$

    where (7) and (8) were explained in the previous case, the inequality (9) combines the inductive hypothesis with Lemma 1 for loops, and the identity (10) is the deletion-contraction formula. □

Combining Lemmas 2 and 3, we proved the following:

Theorem 4 *The simple graph* $X(n,i)$ *is the strongest in* $\Omega_p(n, n-1+i)$ *for all* $i \in \{0,1,2,3,4\}$.

3 | CONCLUSIONS

A simple proof that the strongest simple graphs are the strongest pseudographs was given for $e \leq n+3$. This proof unifies and extends the previous results given in [4, 6]. The main ingredients were the self similarity property shared among the strongest simple graphs, a deletion-contraction formula for pseudographs, the sum-rule and an inductive reasoning over a finite set. This approach can be easily extended to the cases where $e \leq n+7$ if the conjecture posed by Ath and Sobel holds [1], where the authors construct a candidate sequences of uniformly most reliable graphs that satisfy the self similarity property. The general multigraph conjecture is still open.

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